# BEAMS

# FINITE ELEMENT METHODS (17 ME 61)

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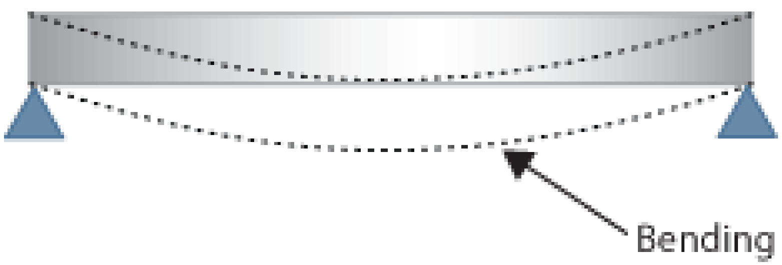
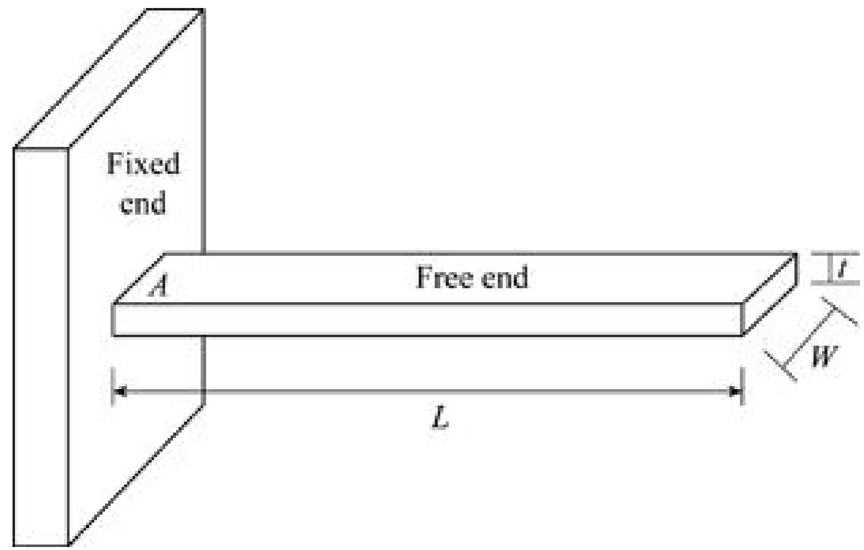
# BEAMS

- Beams play a significant role in engineering applications, including buildings, bridges, automobiles, and aircraft structures.
- A beam is defined as a long, slender structural member whose cross-sectional dimensions are relatively small compared to its length.
- Beams are generally subjected to transverse loading and produce a significant bending moment.
- The deformed shape of a beam is described by vertical displacement ( deflection) and rotation (slope) of the beam.

Simply Supported Beam



# HERMITE SHAPE FUNCTIONS FOR BEAMS

- The shape functions chosen for beam element should meet the $C^1$ continuity requirement which states that the transverse displacements and slopes must be continuous over the element.
- ***Hermite shape functions*** are used for interpolation of dependent variable (displacement) and its derivative (slope).
- The transverse displacement field is of cubic order expressed in natural coordinate system as;

$$v(\xi) = H_1 v_1 + H_2 \left( \frac{\partial v}{\partial \xi} \right)_1 + H_3 v_2 + H_4 \left( \frac{\partial v}{\partial \xi} \right)_2$$

$$Or \quad v(\xi) = H_1 v_1 + H_2\theta_1 + H_3 v_2 + H_4\theta_2 \quad \cdots\cdots(i) \text{ where}$$

*$H_1$ represents deflection at node 1, $H_2$ represents slope at node 1*

*$H_3$ represents deflection at node 2, $H_4$ represents slope at node 2*

$$\left(\frac{\partial v}{\partial x}\right)_1 = \theta_1 = \textit{Slope or rotation at node 1,}$$

$$\left(\frac{\partial v}{\partial x}\right)_2 = \theta_2 = \textit{Slope or rotation at node 2}$$

*$v_1$ = Transverse displacement at node 1,*

*$v_2$ = Transverse displacement at node 2*

$$\text{Also } x = N_1 x_1 + N_2 x_2 = \left(\frac{1-\xi}{2}\right)x_1 + \left(\frac{1+\xi}{2}\right)x_2$$

$$x = \left(\frac{x_1 + x_2}{2}\right) + \left(\frac{x_2 - x_1}{2}\right)\xi \Rightarrow \frac{\partial x}{\partial \xi} = \left(\frac{l_e}{2}\right) \text{ where } l_e = (x_2 - x_1)$$

By chain rule of differentiation, $\frac{\partial v}{\partial \xi} = \frac{\partial v}{\partial x} \times \frac{\partial x}{\partial \xi} = \frac{\partial v}{\partial x}\left(\frac{l_e}{2}\right) = \theta\left(\frac{l_e}{2}\right)$

$$\therefore v(\xi) = H_1 v_1 + \left(\frac{l_e}{2}\right) H_2 \theta_1 + H_3 v_2 + \left(\frac{l_e}{2}\right) H_4 \theta_2 \Rightarrow v = [H]\{q\}$$

$$\therefore [H] = \left[ H_1 \quad \frac{l_e}{2} H_2 \quad H_3 \quad \frac{l_e}{2} H_4 \right] \rightarrow \textit{Hermite shape function}$$

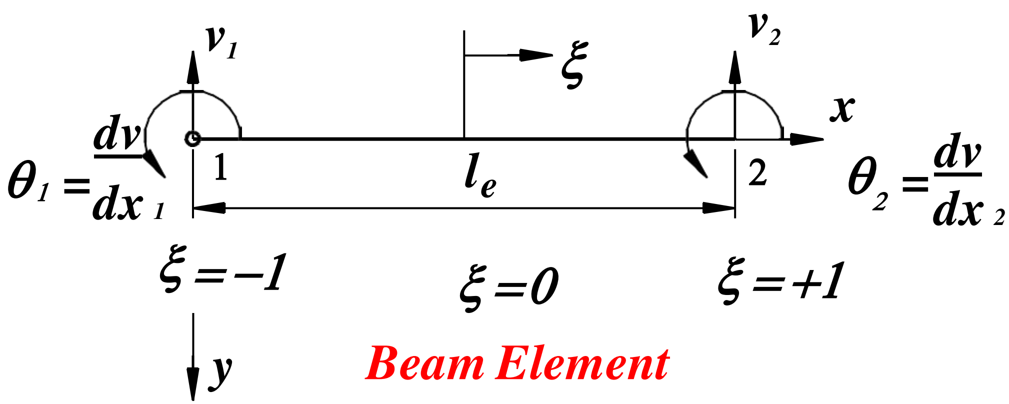
$$\textit{and } \{q\} = \{v_1 \quad \theta_1 \quad v_2 \quad \theta_2\} \rightarrow \textit{Nodal displacement vector}$$

***Boundary condition table for two noded beam element :***

***(To find the values of $H_1, H_2, H_3$ & $H_4$)***

| | $H_1$ | $\frac{\partial H_1}{\partial \xi}$ | $H_2$ | $\frac{\partial H_2}{\partial \xi}$ | $H_3$ | $\frac{\partial H_3}{\partial \xi}$ | $H_4$ | $\frac{\partial H_4}{\partial \xi}$ |
|---|---|---|---|---|---|---|---|---|
| $\xi = -1$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $\xi = +1$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |

# *Derivation of Hermite shape functions of beam element :*




***Beam Element***

***To find shape function $H_1$*** : As there are four degrees of freedom, assume a polynomial displacement model as $\boldsymbol{H_1 = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3}$ where; $a_0$, $a_1$, $a_2$ & $a_3$ are the generalized coordinates to be determined from the nodal coordinates.

***At node 1 :*** $H_1 = 1,\ \xi = -1 \Rightarrow$ $a_o - a_1 + a_2 - a_3 = 1$ *(i)*

Also $\dfrac{\partial H_1}{\partial \xi} = a_1 + 2a_2\xi + 3a_3\xi^2$ At node 1 ($\xi = -1$), $\dfrac{\partial H_1}{\partial \xi} = 0$

$$\Rightarrow a_1 - 2a_2 + 3a_3 = 0 \cdots (ii)$$

***At node 2 :*** $H_1 = 0,\ \xi = 1 \Rightarrow \quad a_o + a_1 + a_2 + a_3 = 0 \ \cdots (iii)$

Also $\dfrac{\partial H_1}{\partial \xi} = a_1 + 2a_2\xi + 3a_3\xi^2$ At node 2 ($\xi = 1$), $\quad \dfrac{\partial H_1}{\partial \xi} = 0$

$\Rightarrow a_1 + 2a_2 + 3a_3 = 0 \cdots (iv)$

Adding (i) & (iii) , $1 = 2a_0 + 2a_2 \Rightarrow a_o + a_2 = 0.5$

Adding (ii) & (iv) , $0 = 2a_1 + 6a_3 \Rightarrow a_1 + 3a_3 = 0 \Rightarrow a_1 = -3a_3$

Substituting the above in Eqn (iii), $a_o + a_1 + a_2 + a_3 = 0$

$-3a_3 + 0.5 + a_3 = 0$ or $a_3 = \left(\dfrac{1}{4}\right) \quad \therefore a_1 = \left(\dfrac{-3}{4}\right)$ Substituting in Eqn (iii)

$(0.5 - a_2) - 0.75 + a_2 + 0.25 = 0,\ a_2 = 0 \quad \therefore a_0 = 0.5$

$\therefore$ ***Shape function*** $H_1 = \dfrac{1}{2} - \dfrac{3}{4}\xi + 0 + \dfrac{1}{4}\xi^3 = \dfrac{1}{4}\left(2 - 3\xi + \xi^3\right)$

Or $\quad H_1 = \dfrac{1}{4}(1 - \xi)^2 (2 + \xi)$

$$H_1 = \frac{1}{4}\left(2 - 3\xi + \xi^3\right) = \frac{1}{4}(1-\xi)^2(2+\xi)$$

***Similarly,***

***using bc's @*** $\xi = -1,\ H_2 = 0,\ \frac{\partial H_2}{\partial \xi} = 1$ ***&*** ***@*** $\xi = +1,\ H_2 = 0,\ \frac{\partial H_2}{\partial \xi} = 0$ ***we get***

$$a_o = \frac{1}{4},\ a_1 = \frac{-1}{4}, a_2 = \frac{-1}{4},\ a_3 = \frac{1}{4} \quad \therefore\ H_2 = \frac{1}{4}\left(1 - \xi - \xi^2 + \xi^3\right) = \frac{1}{4}(1-\xi)^2(\xi+1)$$

***using bc's @*** $\xi = -1,\ H_3 = 0,\ \frac{\partial H_3}{\partial \xi} = 0$ ***&*** ***@*** $\xi = +1,\ H_3 = 1,\ \frac{\partial H_3}{\partial \xi} = 0$ ***we get***

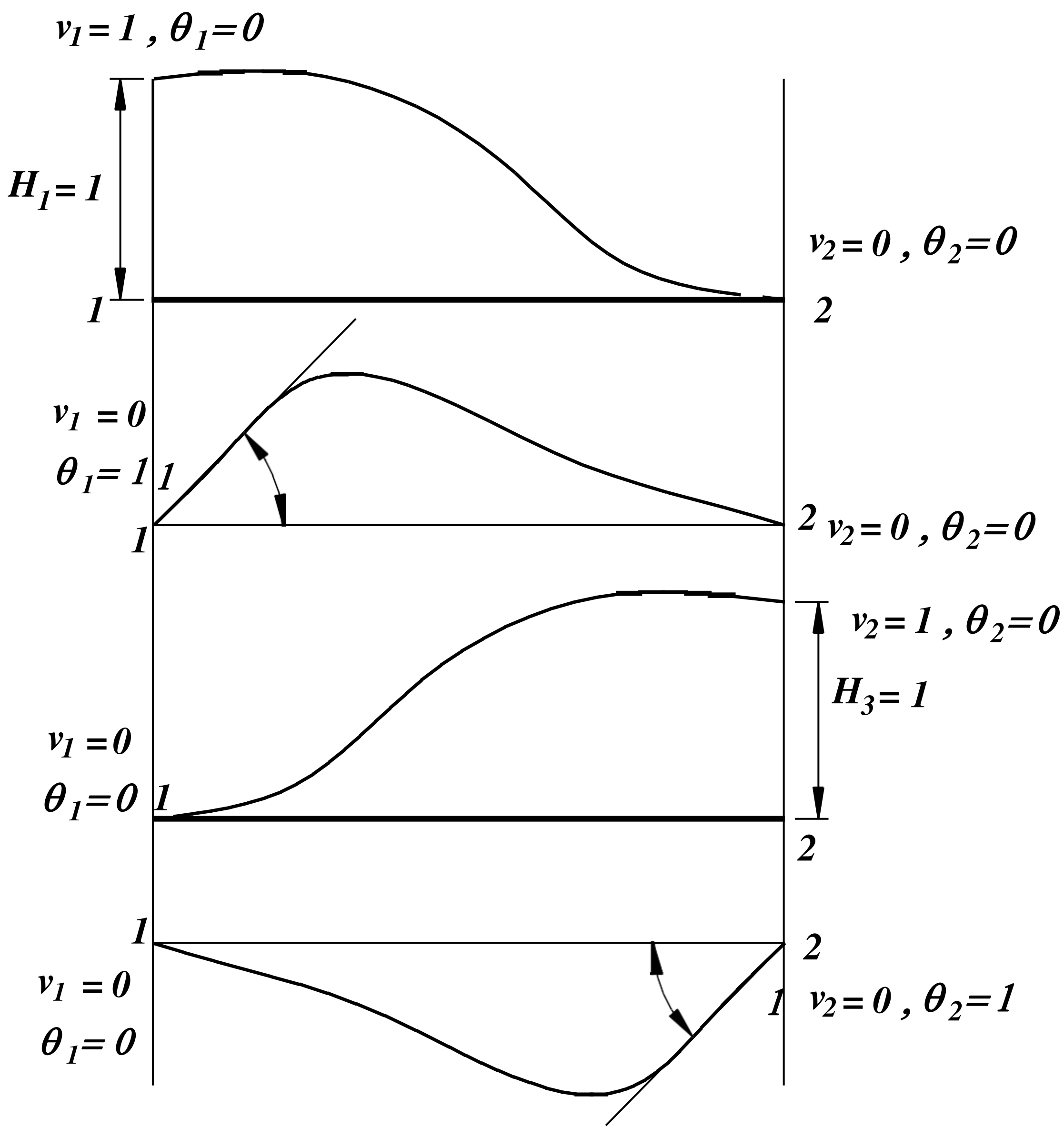
$$a_o = \frac{1}{2},\ a_1 = \frac{3}{4}, a_2 = 0,\ a_3 = \frac{-1}{4} \quad \therefore\ H_3 = \frac{1}{4}\left(2 + 3\xi - \xi^3\right) = \frac{1}{4}(1+\xi)^2(2-\xi)$$

***using bc's @*** $\xi = -1,\ H_4 = 0,\ \frac{\partial H_4}{\partial \xi} = 0$ ***&*** ***@*** $\xi = +1,\ H_4 = 0,\ \frac{\partial H_4}{\partial \xi} = 1$ ***we get***

$$a_o = \frac{-1}{4},\ a_1 = \frac{-1}{4}, a_2 = \frac{1}{4},\ a_3 = \frac{1}{4}\ \&\ H_4 = \frac{1}{4}\left(-1 - \xi + \xi^2 + \xi^3\right) = \frac{1}{4}(1+\xi)^2(\xi-1)$$

As $[H] = \begin{bmatrix} H_1 & \frac{l_e}{2} H_2 & H_3 & \frac{l_e}{2} H_4 \end{bmatrix},$

$$[H] = \begin{bmatrix} \frac{1}{4}\left(2-3\xi+\xi^3\right) \\ \frac{l_e}{8}\left(1-\xi-\xi^2+\xi^3\right) \\ \frac{1}{4}\left(2+3\xi-\xi^3\right) \\ \frac{l_e}{8}\left(-1-\xi+\xi^2+\xi^3\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(1-\xi)^2(2+\xi) \\ \frac{l_e}{8}(1-\xi)^2(\xi+1) \\ \frac{1}{4}(1+\xi)^2(2-\xi) \\ \frac{l_e}{8}(1+\xi)^2(\xi-1) \end{bmatrix}$$




Variation of Hermite Shape functions

## *Derivation of stiffness matrix of a beam element :*

Strain energy for a beam element is given by $U_e = \frac{EI}{2}\int_{l_e}\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx$

As $\frac{\partial v}{\partial \xi} = \frac{\partial v}{\partial x}\times\frac{\partial x}{\partial \xi} \Rightarrow \frac{\partial v}{\partial x} = \left(\frac{2}{l_e}\right)\times\frac{\partial v}{\partial \xi}$ $\left[\because \frac{\partial x}{\partial \xi} = \left(\frac{l_e}{2}\right)\right]$

$\Rightarrow \frac{\partial^2 v}{\partial x^2} = \left(\frac{4}{l_e^2}\right)\times\frac{\partial^2 v}{\partial \xi^2}$ Also, $v = H_1 v_1 + \left(\frac{l_e}{2}\right)H_2\theta_1 + H_3 v_2 + \left(\frac{l_e}{2}\right)H_4\theta_2$

$$\therefore \frac{\partial^2 v}{\partial x^2} = \left(\frac{4}{l_e^2}\right)\times\frac{\partial^2}{\partial \xi^2}\left\{H_1 v_1 + \left(\frac{l_e}{2}\right)H_2\theta_1 + H_3 v_2 + \left(\frac{l_e}{2}\right)H_4\theta_2\right\}$$

$$\frac{\partial^2 v}{\partial x^2} = \left(\frac{4}{l_e^2}\right)\left\{\left(\frac{\partial^2 H_1}{\partial \xi^2}\right)v_1 + \left(\frac{l_e}{2}\right)\theta_1\left(\frac{\partial^2 H_2}{\partial \xi^2}\right) + \left(\frac{\partial^2 H_3}{\partial \xi^2}\right)v_2 + \left(\frac{l_e}{2}\right)\theta_2\left(\frac{\partial^2 H_4}{\partial \xi^2}\right)\right\}$$

$$\frac{\partial^2 v}{\partial x^2} = \left(\frac{4}{l_e^2}\right)\left\{\left[\left(\frac{\partial^2 H_1}{\partial \xi^2}\right)v_1 + \left(\frac{l_e}{2}\right)\theta_1\left(\frac{\partial^2 H_2}{\partial \xi^2}\right) + \left(\frac{\partial^2 H_3}{\partial \xi^2}\right)v_2 + \left(\frac{l_e}{2}\right)\theta_2\left(\frac{\partial^2 H_4}{\partial \xi^2}\right)\right]\right\}$$

$Let\ \frac{\partial^2 v}{\partial x^2} = [C]\{v\}$ where;

$$[C] = \left(\frac{4}{l_e^2}\right)\left[\frac{\partial^2 H_1}{\partial \xi^2} \quad \left(\frac{l_e}{2}\right)\left(\frac{\partial^2 H_2}{\partial \xi^2}\right) \quad \frac{\partial^2 H_3}{\partial \xi^2} \quad \left(\frac{l_e}{2}\right)\left(\frac{\partial^2 H_4}{\partial \xi^2}\right)\right]$$

Substituting in equation of Strain energy, $U_e = \frac{1}{2}\int_{l_e} EI\left([C]\{v\}\right)^2 dx$

$$U_e = \frac{1}{2}\int_{l_e}\{v\}^T\left(EI\,[C]^T[C]\right)\{v\}\,dx = \frac{1}{2}\int_{l_e}\{v\}^T\left[k^e\right]\{v\}\,dx \quad \text{where;}$$

$$\left[k^e\right] = \int_{l_e} EI\,[C]^T[C]\,dx = \int_{-1}^{+1} EI\,[C]^T[C]\frac{l_e}{2}\,d\xi = \frac{l_e EI}{2}\int_{-1}^{+1}[C]^T[C]\,d\xi$$

$$\left[\because dx = \frac{l_e}{2}d\xi \text{ \& limits are } \xi = -1 \text{ \& } +1\right]$$

$$Also\ [C] = \left(\frac{4}{l_e^2}\right)\left[\frac{\partial^2 H_1}{\partial \xi^2} \quad \left(\frac{l_e}{2}\right)\left(\frac{\partial^2 H_2}{\partial \xi^2}\right) \quad \frac{\partial^2 H_3}{\partial \xi^2} \quad \left(\frac{l_e}{2}\right)\left(\frac{\partial^2 H_4}{\partial \xi^2}\right)\right]$$

Substituting $H_1 = \frac{1}{4}\left(2 - 3\xi + \xi^3\right)$, $H_2 = \frac{1}{4}\left(1 - \xi - \xi^2 + \xi^3\right)$,

$H_3 = \frac{1}{4}\left(2 + 3\xi - \xi^3\right)$, $H_4 = \frac{1}{4}\left(2 + 3\xi - \xi^3\right) = \frac{1}{4}(1+\xi)^2(\xi - 1)$ &

differentiating twice, we get;

$$[C] = \left(\frac{1}{l_e^2}\right)\left[6\xi \quad -l_e(1-3\xi) \quad -6\xi \quad l_e(1+3\xi)\right]$$

$$\Rightarrow \int_{-1}^{1} [C]^T [C] d\xi = \frac{2}{l_e^4}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$But\ \left[k^e\right]=\frac{l_e EI}{2}\int_{-1}^{+1}[C]^T[C]\,l_e\,d\xi$$

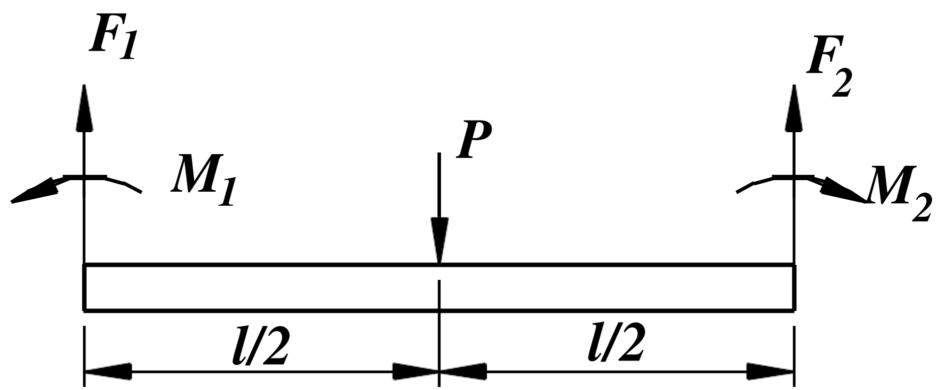
$$\Rightarrow\left[k^e\right]=\frac{l_e EI}{2}\times\frac{2}{l_e^{\,4}}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^{\,2} \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$\left[k^e\right]=\frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^{\,2} \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}\rightarrow$$ ***Stiffness matrix of beam element***

**Load vectors in Beam element :**

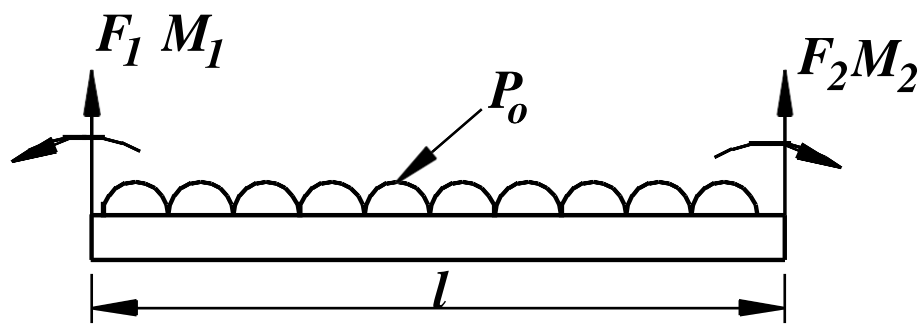
***(i) Load vector due to Point loads :***

$$\{F\} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{P}{2} \\ \dfrac{Pl}{8} \\ \dfrac{P}{2} \\ -\dfrac{Pl}{8} \end{Bmatrix}$$




***(ii) Load vector due to UDL :***

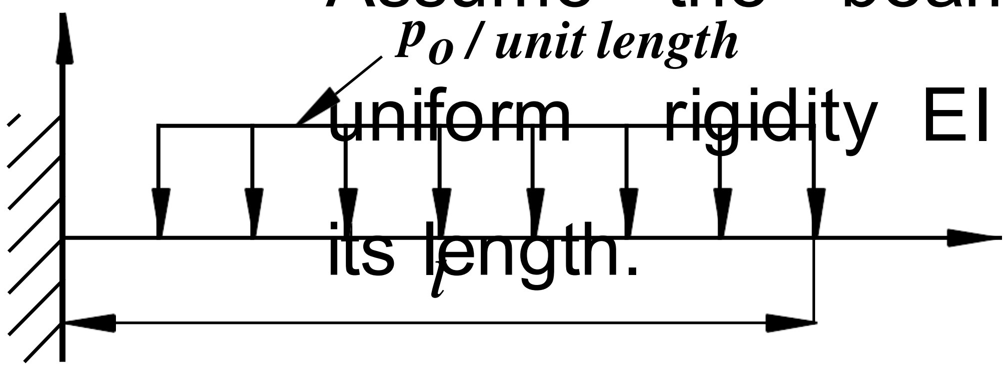
$$\{F\} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{P_o l}{2} \\ \dfrac{P_o l^2}{12} \\ \dfrac{P_o l}{2} \\ -\dfrac{P_o l^2}{12} \end{Bmatrix}$$




***Note : Substitute P or $P_o$ as negative if the loads are downward***

## Problem 1

A cantilever beam subjected to a uniform load qo is as shown in fig. Solve for displacement, rotation & nodal forces. Assume the beam to have uniform rigidity EI throughout its length.



***Stiffness matrix*** : $[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$

***Equilibrium Equation*** : $[K]\{q\} = \{F\}$ where;

$\{q\} = \{v_1 \quad \theta_1 \quad v_2 \quad \theta_2\}^T$ and $\{F\} = \left\{\frac{P_o l}{2} \quad \frac{P_o l^2}{12} \quad \frac{P_o l}{2} \quad -\frac{P_o l^2}{12}\right\}^T$

Using bc's at nodes 1, $v_1 = \theta_1 = 0$ and also $UDL = -P_o$ (as it is downwards)

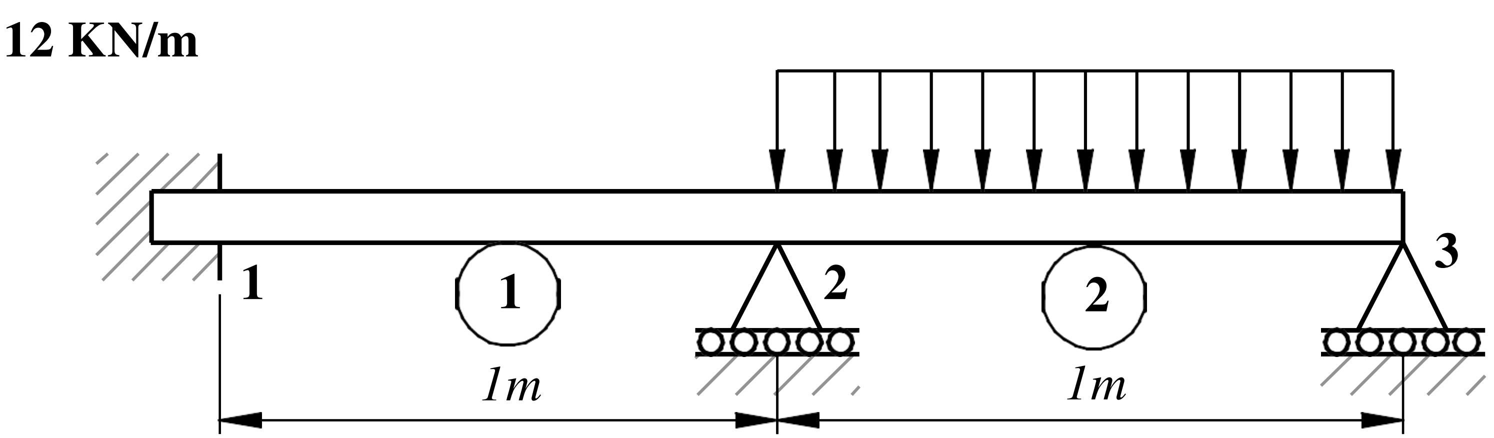
$$\Rightarrow \frac{EI}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l_2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{p_o l}{2} \\ \frac{p_o l^2}{12} \end{Bmatrix} \quad \therefore \quad \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{p_o l^4}{8EI} \\ -\frac{p_o l^3}{6EI} \end{Bmatrix}$$

## *Internal nodal forces:*

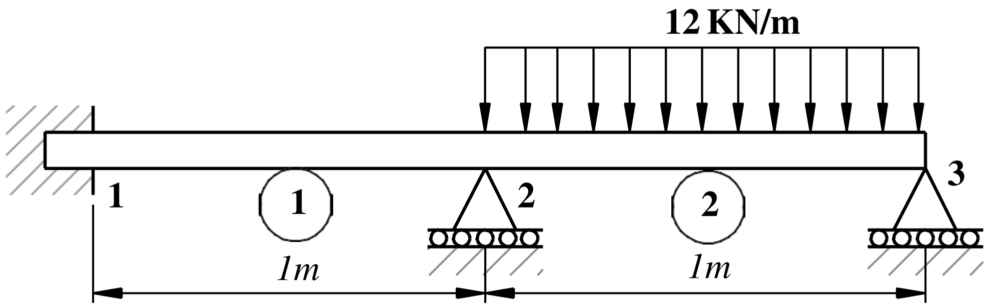
$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = [K]\{q\} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -\dfrac{p_o l^4}{8EI} \\ -\dfrac{p_o l^3}{6EI} \end{Bmatrix} = \begin{Bmatrix} \dfrac{p_o l}{2} \\ -\dfrac{5 p_o l^2}{12} \\ -\dfrac{p_o l}{2} \\ \dfrac{p_o l^2}{12} \end{Bmatrix}$$

# Problem 2

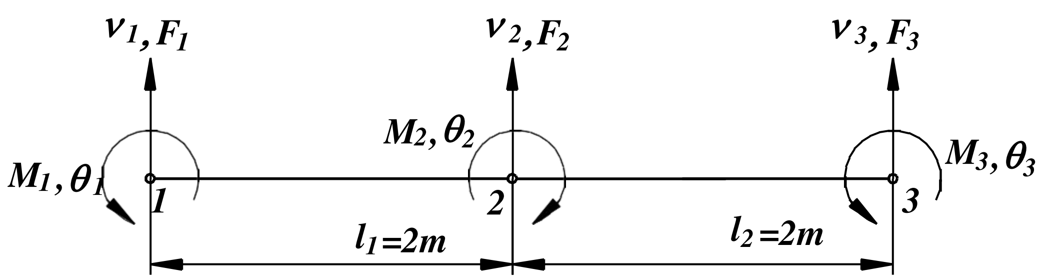
For the beam loaded as shown in fig, determine the slopes at 2 & 3, and the deflection at the midpoint of the distributed load.




E=200 GPa, I = 4 x $10^6$ $mm^4$




E=200 GPa, I = 4 x $10^6$ mm$^4$




**Stiffness matrix** : $[K]=\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$

For element 1,

$$\frac{EI}{l_1^3}=\frac{200\times10^9\times4\times10^{-6}}{1^3}=8\times10^5 N-m,\ 6l_1=6, 4l_1^2=4, 2l_1^2=2$$

$$\left[k^{(1)}\right]=8\times10^5\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & -6 & 12 & -6\\ 6 & 2 & -6 & 4\end{bmatrix}\begin{matrix}v_1\\ \theta_1\\ v_2\\ \theta_2\end{matrix}\qquad \left\{F^{(1)}\right\}=\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}\begin{matrix}v_1\\ \theta_1\\ v_2\\ \theta_2\end{matrix}$$

(columns: $v_1$, $\theta_1$, $v_2$, $\theta_2$)

For element 2, $(P_o=-12\ KN/m\ as\ UDL\ is\ downward)$

$$\frac{EI}{l_2^3}=8\times10^5 N-m,\ 6l_2=6, 4l_2^2=4, 2l_2^2=2, \frac{P_o l_2}{2}=-6\times10^3 N, \frac{P_o l_2^2}{12}=-10^3 Nm$$

$$\left[k^{(2)}\right]=8\times10^5\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & -6 & 12 & -6\\ 6 & 2 & -6 & 4\end{bmatrix}\begin{matrix}v_2\\ \theta_2\\ v_3\\ \theta_3\end{matrix}\qquad \left\{F^{(2)}\right\}=10^3\begin{Bmatrix}-6\\-1\\-6\\1\end{Bmatrix}\begin{matrix}v_2\\ \theta_2\\ v_3\\ \theta_3\end{matrix}$$

(columns: $v_2$, $\theta_2$, $v_3$, $\theta_3$)

After assembling *the* element stiffness *matrices and load* vectors, *we* get *the global stiffness matrix & thereby equilibrium equation.*

$$8\times10^5\begin{bmatrix}12 & 6 & -12 & 6 & 0 & 0\\ 6 & 4 & -6 & 2 & 0 & 0\\ -12 & -6 & 24 & 0 & -12 & 6\\ 6 & 2 & 0 & 8 & -6 & 2\\ 0 & 0 & -12 & -6 & 12 & -6\\ 0 & 0 & 6 & 2 & -6 & 4\end{bmatrix}\begin{Bmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\\ v_3\\ \theta_3\end{Bmatrix}=10^3\begin{Bmatrix}0\\ 0\\ -6\\ -1\\ -6\\ 1\end{Bmatrix}$$

*Applying boundary conditions* $\boldsymbol{v_1 = \theta_1 = 0}$ *(fixed end )* ***and*** $\boldsymbol{v_2 = v_3 = 0}$ *(vertical movements are constrained at roller supports) the equilibrium equation becomes;*

$$8\times10^5\begin{Bmatrix}8 & 2\\ 2 & 4\end{Bmatrix}\begin{Bmatrix}\theta_2\\ \theta_3\end{Bmatrix}=\begin{Bmatrix}-1000\\ +1000\end{Bmatrix}\Rightarrow \theta_2=-2.68\times10^{-4}\ rad,\ \theta_3=4.464\times10^{-4}\ rad$$

***Vertical deflection at mid span (of element 2) is*** **=** *Hv at* $\xi = 0$

But, $\boldsymbol{v} = H_1 v_1 + \frac{l_e}{2} H_2 \theta_2 + H_3 v_2 + \frac{l_e}{2} H_4 \theta_3$ $\quad Here, v_1 = v_2 = 0$

$$\Rightarrow \boldsymbol{v} = 0 + \frac{l_e}{2} H_2 \theta_2 + 0 + \frac{l_e}{2} H_4 \theta_3$$

$Also\ H_2 = \frac{1}{4}\left(1 - \xi - \xi^2 + \xi^3\right) \therefore\ At\ \xi = 0, H_2 = \frac{1}{4}$

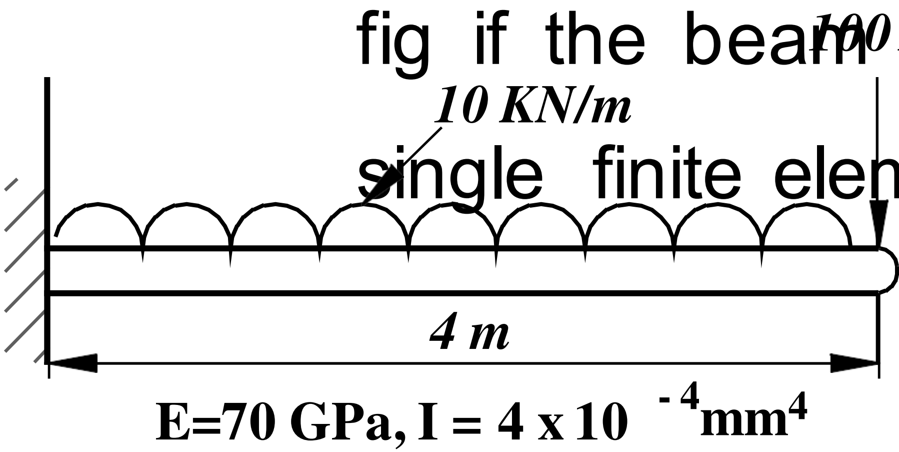
$and\ H_4 = \frac{1}{4}\left(-1 - \xi + \xi^2 + \xi^3\right) \quad \therefore At\ \xi = 0, H_2 = \frac{1}{4}$

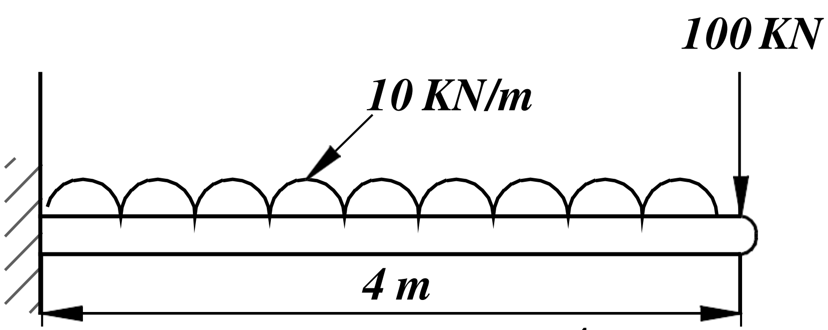
$$\boldsymbol{v} = 0 + \frac{1}{2} \times \left(\frac{1}{4}\right) \times \left(-2.68 \times 10^{-4}\right) + 0 + \frac{1}{2} \times \left(-\frac{1}{4}\right)\left(4.464 \times 10^{-4}\right)$$

$\boldsymbol{v} = -0.893 \times 10^{-4}\, m =$ ***0.0893 mm***

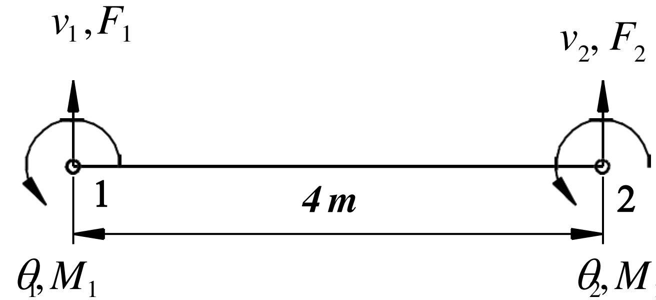
# Problem 3

Determine the maximum deflection & the internal loads in the uniform cross section of the cantilever beam shown in fig if the beam is treated as a single finite element.






E=70 GPa, I = 4 x 10 $^{-4}$mm$^4$



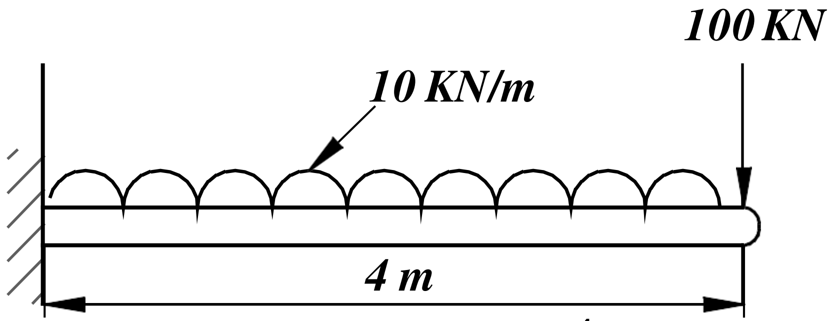

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

*Here,* $\dfrac{EI}{l^3} = \dfrac{70\times10^9\times4\times10^{-4}}{4^3} = 0.4375\times10^6$

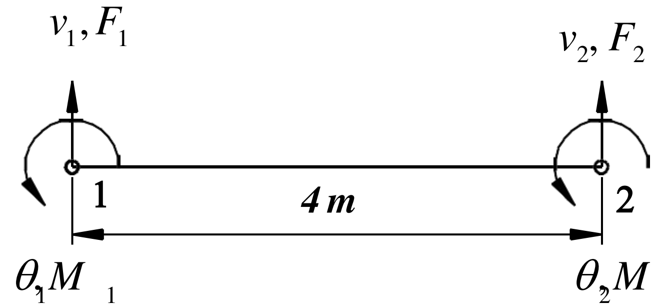
$$6l = 6\times4 = 24,\ \ 4l^2 = 4\times4^2 = 64,\ \ 2l^2 = 2\times4^2 = 32$$

$$\Rightarrow [k] = 0.4375\times10^6 \begin{bmatrix} 12 & 24 & -12 & 24 \\ 24 & 64 & -24 & 32 \\ -12 & -24 & 12 & -24 \\ 24 & 32 & -24 & 64 \end{bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix}$$

(columns: $v_1$, $\theta_1$, $v_2$, $\theta_2$)




E=70 GPa, I = 4 x 10 $^{-4}$mm$^4$




$$\{F\} = \begin{Bmatrix} \dfrac{-q_o l}{2} \\ \dfrac{-q_o l^2}{12} \\ \dfrac{-q_o l}{2} \\ \dfrac{q_o l^2}{12} \end{Bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix} = \begin{Bmatrix} \dfrac{-10\times 4}{2} \\ \dfrac{-10\times 4^2}{12} \\ \dfrac{-10\times 4}{2} - 100 \\ \dfrac{10\times 4^2}{12} \end{Bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix} = \begin{Bmatrix} \textbf{-20 KN} \\ \textbf{-13.33 KN - m} \\ \textbf{-120 KN - m} \\ \textbf{13.33 KN - m} \end{Bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix}$$

***The equilibrium* equation is** $[K]\{q\}=\{F\}$

$$\Rightarrow 0.4375\times10^{6}\begin{bmatrix} 12 & 24 & -12 & 24 \\ 24 & 64 & -24 & 32 \\ -12 & -24 & 12 & -24 \\ 24 & 32 & -24 & 64 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -87.62\times10^{-3} \\ 32.38\times10^{-3} \end{Bmatrix}=\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix}$$

*The equilibrium equation after applying bc's* $v_1=\theta_1=0$

***and elimination** of corresponding rows & columns is;*

$$0.4375\times10^{6}\begin{bmatrix} 12 & -24 \\ -24 & 64 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix}=10^{3}\begin{Bmatrix} -120 \\ 13.33 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix}=10^{-3}\begin{Bmatrix} -87.62\, m \\ 32.38\, rad \end{Bmatrix}$$

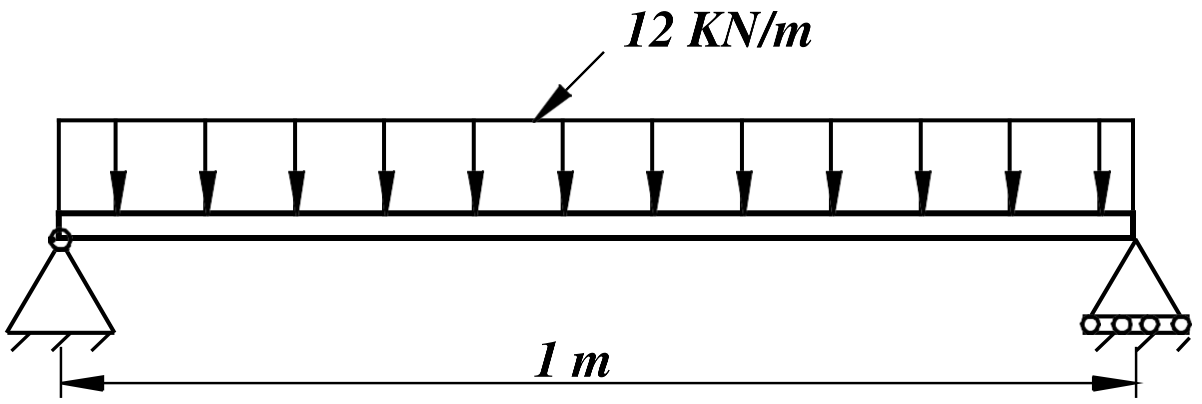
***Internal loads of the elements is* given *by :***

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = [K]\{q\} = 0.4375\times10^6 \begin{bmatrix} 12 & 24 & -12 & 24 \\ 24 & 64 & -24 & 32 \\ -12 & -24 & 12 & -24 \\ 24 & 32 & -24 & 64 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -87.62\times10^{-3} \\ 32.38\times10^{-3} \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} 120KN \\ 466.67KNm \\ -120KN \\ 13.33KNm \end{Bmatrix}$$

## Problem 4

Fig shows a simply supported subjected to a uniformly distributed load. Obtain the slopes at the supports and the maximum deflection in the beam. Take E= 200 Gpa and I = 2 x $10^6$ $mm^4$.



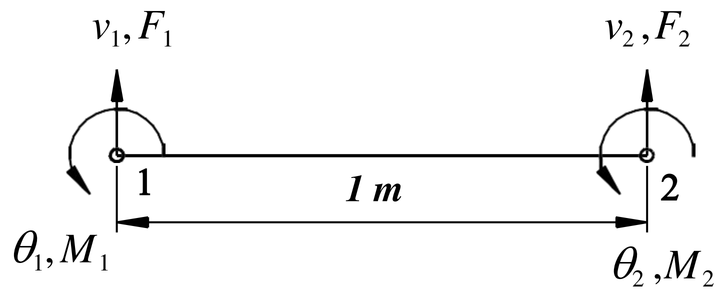
**Stiffness matrix :** $$[K]=\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}$$

**Here,** $$\frac{EI}{l^3}=\frac{200\times10^9\times2\times10^6\times10^{-12}}{1^3}=400\times10^3=400\ KN-m$$

$$6l=6\times1=6,\ 4l^2=4\times1^2=4,\ 2l^2=2\times1^2=2$$

$$[k]=400\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & -6 & 12 & -6\\ 6 & 2 & -6 & 4\end{bmatrix}\begin{matrix}v_1\\ \theta_1\\ v_2\\ \theta_2\end{matrix}\qquad \{q\}=\begin{Bmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\end{Bmatrix}$$

(columns: $v_1$, $\theta_1$, $v_2$, $\theta_2$)




$$Load\ vector\ \{F\} = \left\{ \frac{p_o l}{2} \quad \frac{p_o l^2}{12} \quad \frac{p_o l}{2} \quad -\frac{p_o l^2}{12} \right\}^T \ As\ UDL\ is\ downward,$$

$$p_o = -12\ KN/m,\quad \{F\} = \begin{Bmatrix} -6 \\ -1 \\ -6 \\ 1 \end{Bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix} \text{ Equilibrium equation is } [K]\{q\} = \{F\}$$

$$\Rightarrow 400 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -6 \\ -1 \\ -6 \\ 1 \end{Bmatrix}$$

*The equilibrium equation after applying bc's* $v_1 = v_2 = 0$

***and elimination*** *of corresponding rows & columns is;*

$$400\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}-1\\ +1\end{Bmatrix} \Rightarrow \begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = 10^{-3}\begin{Bmatrix}-1.25\\ +1.25\end{Bmatrix} \textit{radians}$$

***Maximum deflection (at mid span) is*** $= Hv \text{ at } \xi = 0$

But, $\boldsymbol{v} = H_1 v_1 + \frac{l_e}{2} H_2 \theta_1 + H_3 v_2 + \frac{l_e}{2} H_4 \theta_2$ *Here,* $v_1 = v_2 = 0$

$$\Rightarrow \boldsymbol{v} = 0 + \frac{l_e}{2} H_2 \theta_1 + 0 + \frac{l_e}{2} H_4 \theta_2 \quad \textit{Also } H_2 = \frac{1}{4}\left(1 - \xi - \xi^2 + \xi^3\right)$$

$$\therefore \textit{At } \xi = 0, H_2 = \frac{1}{4}$$

$$\text{Also } H_4 = \frac{1}{4}\left(-1 - \xi + \xi^2 + \xi^3\right) \qquad \therefore \text{ At } \xi = 0,\ H_2 = -\frac{1}{4}$$

$$\boldsymbol{v} = 0 + \frac{l_e}{2} H_2 \theta_1 + 0 + \frac{l_e}{2} H_4 \theta_2$$

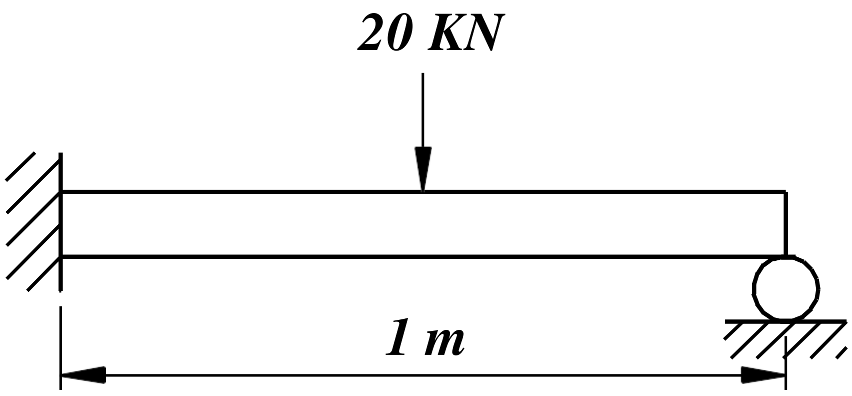
$$\boldsymbol{v} = 0 + \frac{1}{2} \times \left(\frac{1}{4}\right) \times \left(-1.25 \times 10^{-3}\right) + 0 + \frac{1}{2} \times \left(-\frac{1}{4}\right)\left(1.25 \times 10^{-3}\right)$$

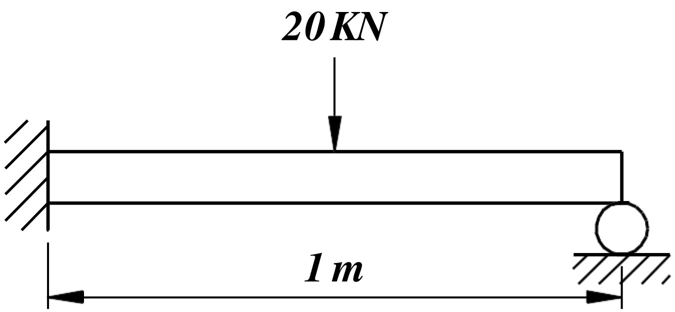
$$v = \left(\frac{-1.25 \times 10^{-3}}{4}\right) = -3.125 \times 10^{-4}\, m = \boldsymbol{0.0003125\, mm}$$

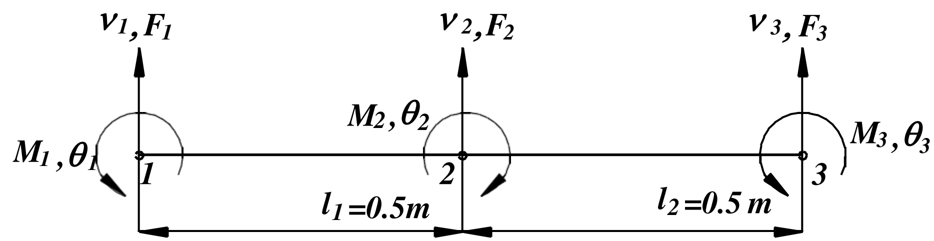
# Problem 5

A uniform cross section beam is fixed at one end supported by a roller at the other end. A concentrated of 20 KN is applied at the midspan of the beam as shown in fig. Determine the slopes at supports & deflection under the load.

Take E= 210 GPa and I = 2500 $mm^4$.






**Stiffness matrix :** $[K] = \dfrac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$

*For element 1,* $\dfrac{EI}{l_1^3} = \dfrac{210\times10^9 \times 2500\times10^{-12}}{0.5^3} = 4200 N-m = 4.2\ KN-m$

$$6l_1 = 6\times0.5 = 3,\ 4l_1^2 = 4(0.5)^2 = 1,\ 2l_1^2 = 2(0.5)^2 = 0.5$$

$$\therefore \left[k^{(1)}\right] = 4.2\begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix}\begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix} = \left[k^{(2)}\right] \text{ (Due to symmetry)}$$

(columns: $v_1$, $\theta_1$, $v_2$, $\theta_2$)

Global stiffness matrix $[K] = [k^{(1)}] + [k^{(2)}]$

$$\Rightarrow [K] = 4.2\begin{bmatrix} 12 & 3 & -12 & 3 & 0 & 0 \\ 3 & 2 & -3 & 0.5 & 0 & 0 \\ -12 & -3 & 24 & 0 & -12 & 3 \\ 3 & 0.5 & 0 & 2 & -3 & 0.5 \\ 0 & 0 & -12 & -3 & 12 & -3 \\ 0 & 0 & 3 & 0.5 & -3 & 1 \end{bmatrix}$$ Using $[K]\{q\} = \{F\}$,

$$i.e.\ 4.2\begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -20 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

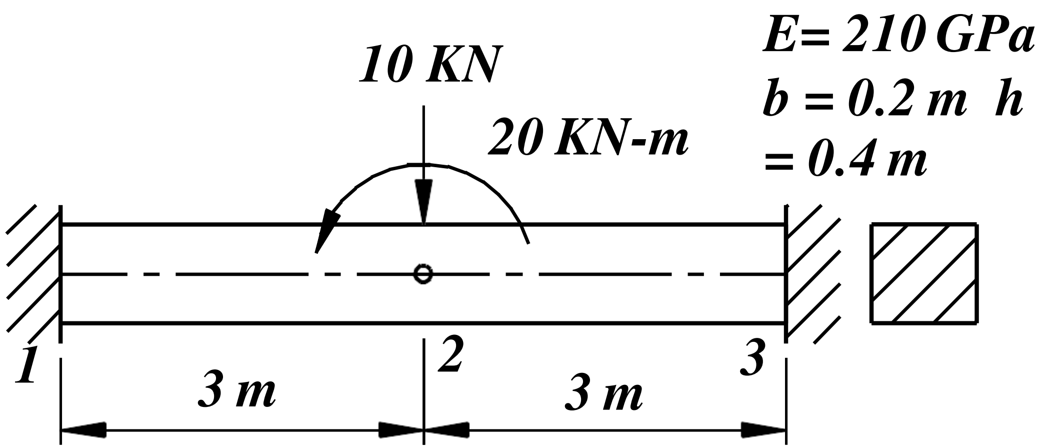
Applying boundary conditions $\boldsymbol{v_1 = \theta_1 = 0}$ (fixed *end* ***) and*** $\boldsymbol{v_3 = 0}$ (vertical *movements* are constrained at roller supports) the *equilibrium equation becomes;*

$$4.2\begin{bmatrix} 24 & 0 & 3 \\ 0 & 2 & 0.5 \\ 3 & 0.5 & 1 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -20 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow v_2 = -0.3472\ m,\ \theta_2 = 0.2976\ rad\ ,\ \theta_3 = 1.19\ rad$$

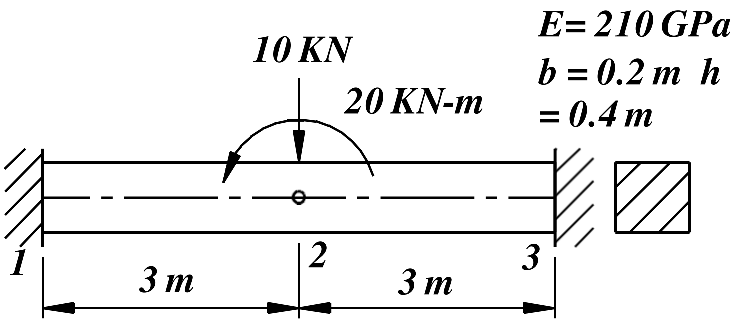
**Problem 6**

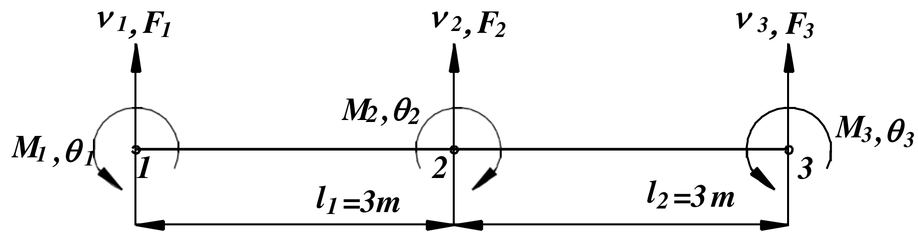
For the beam fixed at both ends and loaded as shown in fig, determine the displacement and slopes at node 2 & reactions at the nodes 1 & 3.




*Here, Moment of inertia of the beam section is*

$$I = \frac{bh^3}{12} = \frac{0.2 \times 0.4^3}{12} = \mathbf{1.067 \times 10^{-3}}$$




**Stiffness matrix :** $[K] = \dfrac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$

*For element 1,* $\dfrac{EI}{l_1^3} = \dfrac{210\times10^9\times1.067\times10^{-3}}{3^3} = 8.3\times10^6\,N-m = \mathbf{8.3\times10^3\,KN}$

$$6l_1 = 6\times3 = 18,\ 4l_1^2 = 4(3)^2 = 36,\ 2l_1^2 = 2(3)^2 = 18$$

$$\therefore [k^{(1)}] = 8.3\times10^3 \begin{bmatrix} 12 & 18 & -12 & 18 \\ 18 & 36 & -18 & 18 \\ -12 & -18 & 12 & -18 \\ 18 & 18 & -18 & 36 \end{bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix} = [k^{(2)}] \text{ (Due to symmetry)}$$

(columns: $v_1$, $\theta_1$, $v_2$, $\theta_2$)

Global stiffness matrix $[K] = [k^{(1)}] + [k^{(2)}]$

$$\Rightarrow [K] = 8.3\times10^{3}\begin{bmatrix} 12 & 18 & -12 & 18 & 0 & 0 \\ 18 & 36 & -18 & 18 & 0 & 0 \\ -12 & -18 & 24 & 0 & -12 & 18 \\ 18 & 18 & 0 & 72 & -18 & 18 \\ 0 & 0 & -12 & -18 & 12 & -18 \\ 0 & 0 & 18 & 18 & -18 & 36 \end{bmatrix}$$

Using $[K]\{q\} = \{F\}$,

$$i.e.\ 8.3\times10^{3}\begin{bmatrix} 12 & 18 & -12 & 18 & 0 & 0 \\ 18 & 36 & -18 & 18 & 0 & 0 \\ -12 & -18 & 24 & 0 & -12 & 18 \\ 18 & 18 & 0 & 72 & -18 & 18 \\ 0 & 0 & -12 & -18 & 12 & -18 \\ 0 & 0 & 18 & 18 & -18 & 36 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -10 \\ 20 \\ 0 \\ 0 \end{Bmatrix}$$

*Applying boundary conditions* $\boldsymbol{v_1 = \theta_1 = v_3 = \theta_3 = 0}$ *(fixed ends)*

*the equilibrium equation becomes;*

$$8.3\times10^{3}\begin{bmatrix} 24 & 0 \\ 0 & 72 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -10 \\ 20 \end{Bmatrix}$$

$$\Rightarrow v_2 = -50.22\times10^{-6}\ m,\ \theta_2 = 33.46\times10^{-6}\ rad.$$

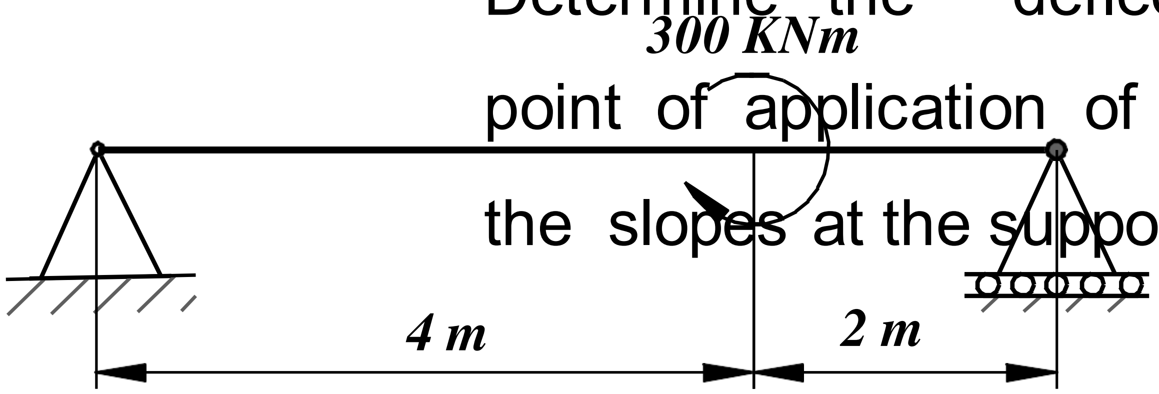
***Reactions :*** $\{R\} = [K]\{Q\} - \{F\}$

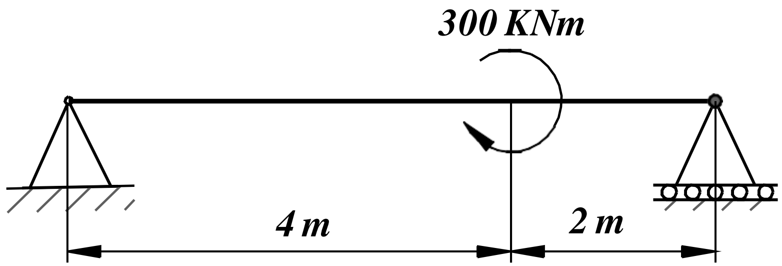
$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{Bmatrix} = 8.3\times10^{3}\begin{bmatrix} 12 & 18 & -12 & 18 & 0 & 0 \\ 18 & 36 & -18 & 18 & 0 & 0 \\ -12 & -18 & 24 & 0 & -12 & 18 \\ 18 & 18 & 0 & 72 & -18 & 18 \\ 0 & 0 & -12 & -18 & 12 & -18 \\ 0 & 0 & 18 & 18 & -18 & 36 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -50.22\times10^{-6} \\ 33.46\times10^{-6} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -10 \\ 20 \\ 0 \\ 0 \end{Bmatrix}$$

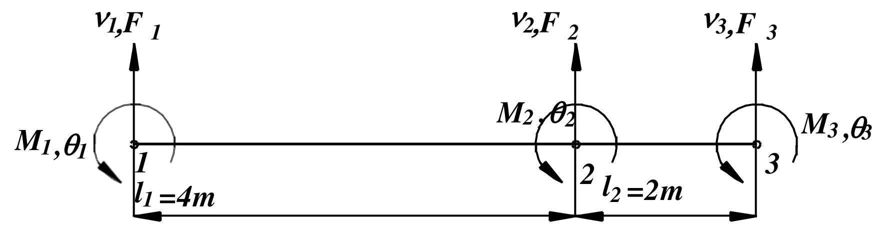
$$\therefore \{R\} = \{10 \quad 12.5 \quad 0 \quad 0 \quad 0 \quad -2.5\}^{T}$$

## Problem 7

A simply supported beam of span 6m and of uniform flexural rigidity EI = 40000 KN-m$^2$ is subjected to a clockwise couple of 300 KNm at a distance of 4m from the left end. Determine the deflections at the point of application of the couple & the slopes at the supports.





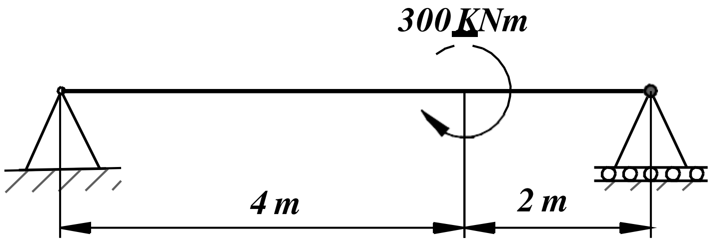



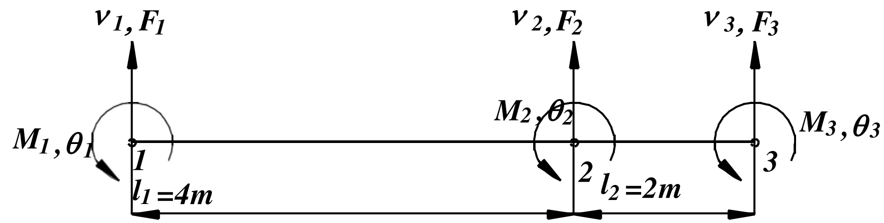

For element 1, stiffness matrix $\left[k^{(1)}\right]=\dfrac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$

$$Here,\ \frac{EI}{l_1^3}=\frac{40000\times10^3}{4^3}=0.625\times10^6,\ 6l_1=24, 4l_1^2=64, 2l_1^2=32$$

$$\left[k^{(1)}\right]=10^6\begin{bmatrix} \overset{v_1}{7.5} & \overset{\theta_1}{15} & \overset{v_2}{} & \overset{\theta_2}{-7.5} \\ & & & 15 \\ -7.5 & 15 & & 40 \\ & -15 & & 20 \\ 15 & & & \\ & -15 & & 7.5 \\ & & & -15 \end{bmatrix}\begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix}$$

Load vector $\left\{F^{(1)}\right\}=\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -300\times10^{-3} \end{Bmatrix}\begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix}$




For element 2, $\dfrac{EI}{l_2^3} = \dfrac{40000\times 10^3}{2^3} = 5\times 10^6,\ 6l_2 = 12, 4l_2^2 = 16, 2l_2^2 = 8$

$$\left[k^{(2)}\right] = 10^6 \begin{bmatrix} 60 & 60 & -60 & 60 \\ 60 & 80 & -60 & 40 \\ -60 & -60 & 60 & -60 \\ 60 & 40 & -60 & 80 \end{bmatrix} \begin{matrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix}$$

(columns: $v_2$, $\theta_2$, $v_3$, $\theta_3$)

$$\left\{F^{(2)}\right\} = \begin{Bmatrix} 0 \\ -300 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-3} \begin{matrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix}$$

After assembling the element stiffness matrices and load vectors, we get

the global stiffness matrix $[K] = k^{(1)} + k^{(2)}$ & thereby the equilibrium equations of the whole beam is given by; $[K]\{v\} = F$

$$10^6 \begin{bmatrix} 7.5 & 15 & -7.5 & 15 & 0 & 0 \\ 15 & 40 & -15 & 20 & 0 & 0 \\ -7.5 & -15 & 67.5 & 45 & -60 & 60 \\ 15 & 20 & 45 & 120 & -60 & 40 \\ 0 & 0 & -60 & -60 & 60 & -60 \\ 0 & 0 & 60 & 40 & -60 & 80 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -300\times10^3 \\ 0 \\ 0 \end{Bmatrix}$$

*Using bc's $v_1 = v_3 = 0$, by elimination approach, we get dispalcements & slopes;*

$$10^6 \begin{bmatrix} 40 & -15 & 20 & 0 \\ -15 & 67.5 & 45 & 60 \\ 20 & 45 & 120 & 40 \\ 0 & 60 & 40 & 80 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -300\times10^3 \\ 0 \\ 0 \end{Bmatrix}$$

By matrix multplication;

From first row, $40\theta_1 - 15v_2 + 20\theta_2 = 0 \quad \cdots\cdots(i)$

From second row, $-15\theta_1 + 67.5v_2 + 45\theta_2 + 60\theta_3 = -0.3 \quad \cdots\cdots(ii)$

From third row, $20\theta_1 + 45v_2 + 120\theta_2 + 40\theta_3 = 0 \quad \cdots\cdots(iii)$

From fourth row, $60v_2 + 40\theta_2 + 80\theta_3 = 0 \quad \cdots\cdots(iv)$

Multiplying eqn (iii) by 2 and subtracting eqn (ii), we get

$$105v_2 + 220\theta_2 + 80\theta_3 = 0 \quad \cdots\cdots(v)$$

Multiplying eqn (iii) by 0.75 and adding eqn (ii), we get

$$101.25v_2 + 135\theta_2 + 90\theta_3 = -0.3 \quad \cdots\cdots(vi)$$

Solving equations $(iv)$, $(v)$ & $(vi)$, we get;

$$\boldsymbol{v_2 = -0.0267\,m,\ \theta_2 = 6.67 \times 10^{-3}\ rad}$$

$$\boldsymbol{\theta_3 = 0.0167\,rad}, \qquad \boldsymbol{\theta_1 = 0.0177\,rad}$$