Module-2

FINITE ELEMENT ANALYSIS

(17 ME 61)

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Module-2

One dimensional finite elements, Bar & Truss elements;

- Linear elements, Principle of minimum potential energy, admissible displacement function, stiffness matrix, strain matrix, static analysis using elimination method, penalty method, boundary conditions and assembly of load vector,
- Convergence and Compatibility conditions, Shape functions for 1D linear, quadratic and Truss elements

Interpolation models

- Interpolation models are defined as the appropriate ***mathematical model or trial function*** which represents the displacement variation within the element.

- The following types of interpolation models are used in Variational methods/FEM.

1. Trigonometric functions
2. Polynomial function

Among the above, polynomial models are most widely used due to ease of formulating, calculating (differentiating & integrating) & better

Polynomial form of interpolation model

A polynomial type of interpolation model assumed to represent the displacement variation within an element, then the dispalcement can for be expressed as ;

$$u(x)= a_0 + a_1 x + a_2 x^2 + a_3 x^3 .... \text{ (for 1-D element)}$$

$$u(x,y)= a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^2 y + a_7 xy^2 + ...$$

(for 2-D element)

If in the above polynomials, terms upto x^1 & y^1 are considered, it is said to be a linear model.

If terms upto x^2 & y^2 are considered, it is said to be a quadratic model & if terms upto x^3 & y^3 are considered, it is said to be a cubic model & so on.

Convergence Criteria

- Convergence implies results obtained by FEA solution reaches the exact solution. It depends on the proper selection of displacement field variable & order of the interpolation polynomials.
- The convergence of the finite element solution can be achieved if the following three conditions are fulfilled by the assumed displacement function.

1. The displacement function must be continuous within the elements. This can be ensured by choosing a suitable polynomial. For example, for

an n degrees of polynomial, displacement function in 1-D problem can be chosen as:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 .... a_n x^n$$

Convergence Criteria....

2.The displacement function must be capable of rigid body displacements of the element. The constant term used in the polynomial (a_0) ensures this condition. (Even for x=0, the displacement will be equal to a_o)

3.The displacement function must include the constant strains states of the element. As element becomes infinitely small, strain should be constant in the element. Hence, the displacement function should include terms for representing constant strain states. The second term used in the polynomial (a_1) ensures this condition. (As differentiation of a_1x will be a_1 , a constant)

Compatibility

- Displacement should be compatible between adjacent elements. There should not be any discontinuity or overlapping when deformed.
- The adjacent elements must deform without causing openings, overlaps or discontinuities between the elements.

Elements which satisfy all the three convergence requirements and compatibility condition are called Compatible or Conforming elements.

Criteria for selection of order of interpolation polynomial

- The number of generalized coordinates should be equal to the number of degrees of freedom of the elements.

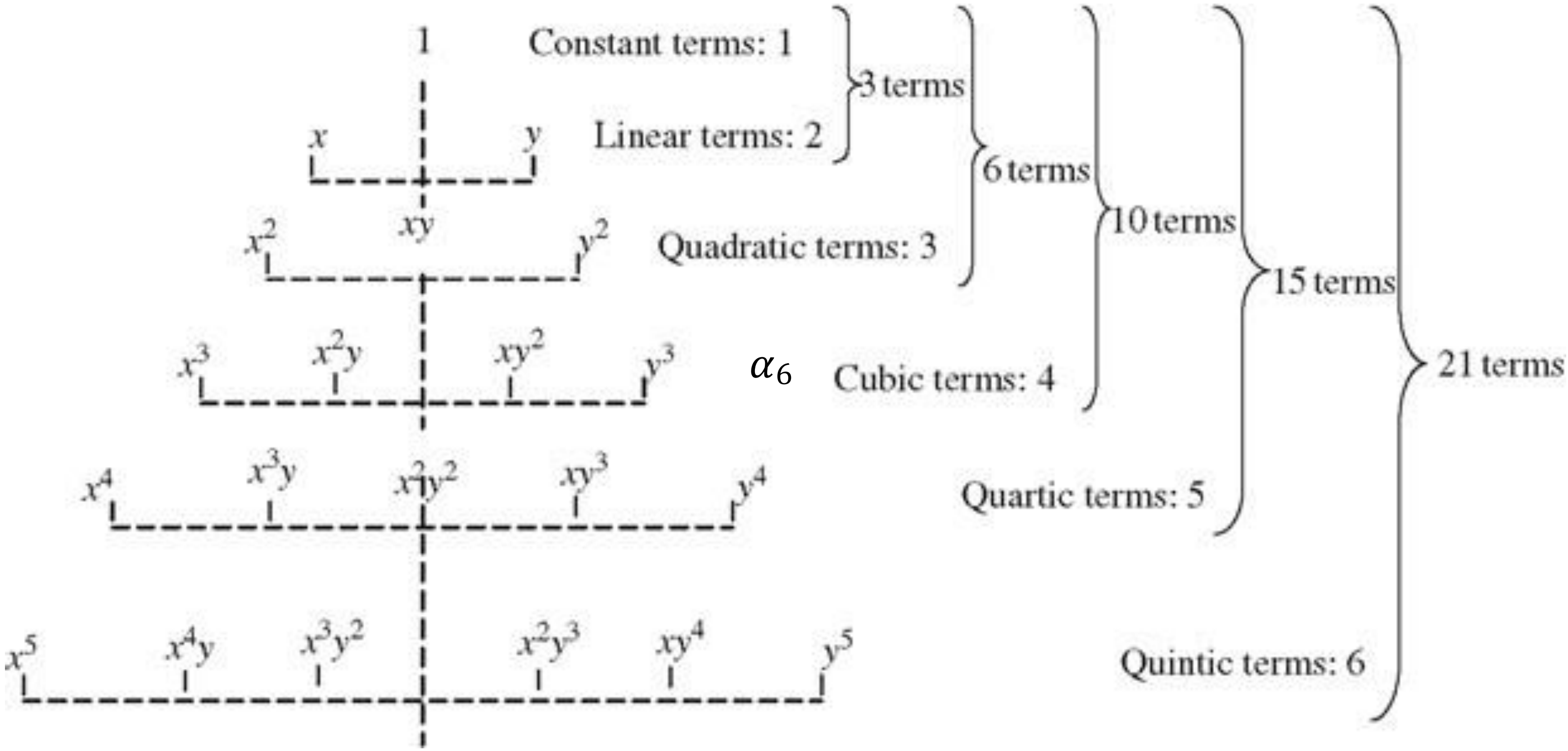
- The pattern of variation of the polynomial should be independent of the local coordinate system. ***(Geometric or spatial isotropy or Geometric invariance).***

- The interpolation polynomial should satisfy the convergence requirements.

- Displacement shape should not change with a change in local coordinate system. This can be achieved if polynomial is balanced in case all terms cannot be completed.

- This „balanced‟ representation can be achieved with the help of Pascal triangle in case of a 2 D polynomial. ***The geometric invariance can be ensured by the selection of the corresponding order of terms on either side of the axis of symmetry.***

Geometric invariance (or Spatial isotropy); Pascal's triangle



Pascal's triangle

Geometric invariance (or isotropy); Pascal's triangle

Ex : If a cubic model is assumed, displacement polynomial using Pascal's triangle is ;

$$U(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^3 + a_7 y^3$$

or

$$U(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^2 y + a_7 xy^2$$

- In the above polynomials, if we interchange x & y terms, the pattern does not change.
- In both the equations, the same variable occur even after interchanging. These polynomials are known as "Balanced Polynomials"

Coordinate systems

- Co ordinate system is a space where configuration of a body is represented.

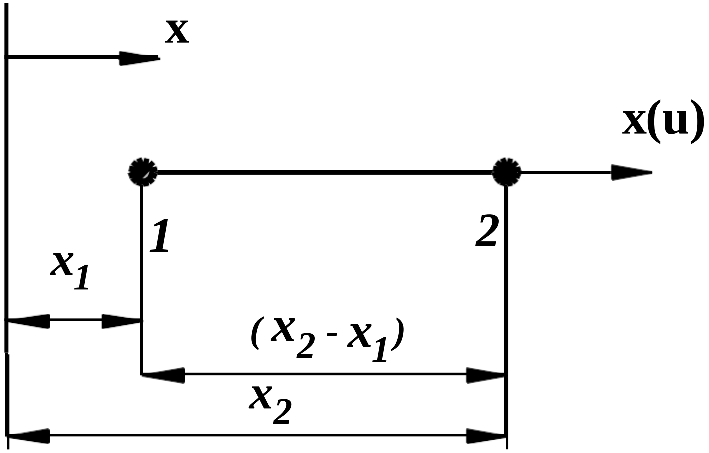
Ex: Cartesian Coordinate system, Polar Coordinate system

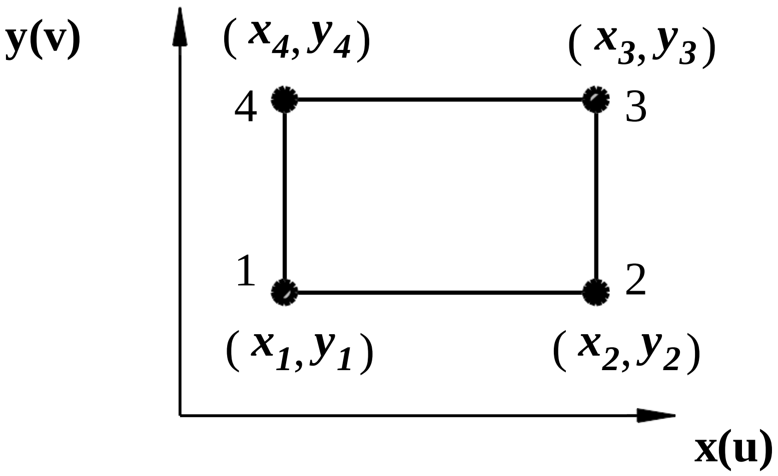
- **In FEM**, these general coordinate systems are further classified as;

1. Global Coordinate system
2. Local Coordinate system
3. Natural coordinate system

Global Coordinate system

- The global coordinate system corresponds to the entire body and used to define the points on the entire body.
- Fig shows method of representation in global coordinate system.

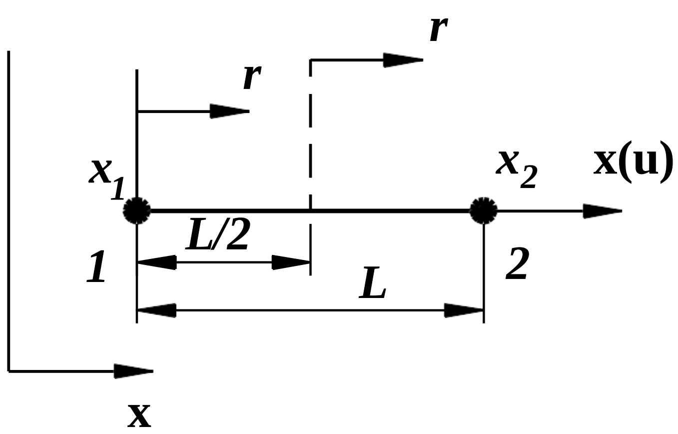




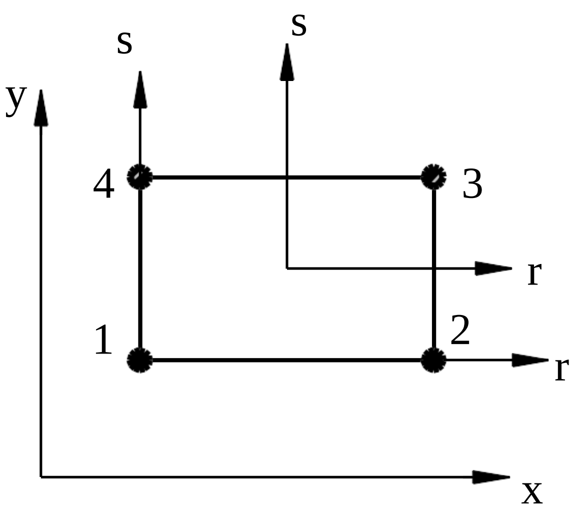
1-D Global coordinate system **2-D Global coordinate system**

Local Coordinate system

- A local coordinates system whose origin is located within the element in order to simplify the algebraic manipulations in the derivation of the element matrix.
- Local coordinate system corresponds to a particular element in the body , and the numbering is done to that particular element neglecting the entire body .



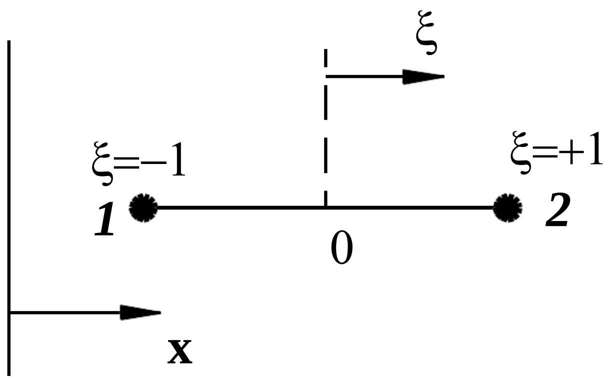
1-D Local coordinate system



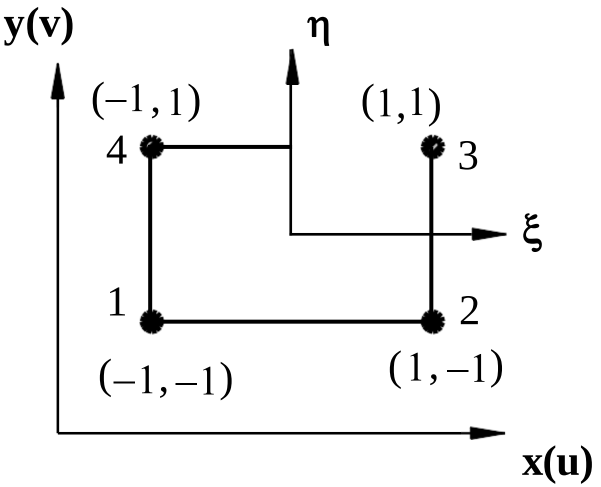
2-D Local coordinate system

Natural Coordinate system

- Natural coordinate system - Similar to local coordinate system but a node is expressed by a dimensionless set of numbers whose magnitude never exceeds unity.

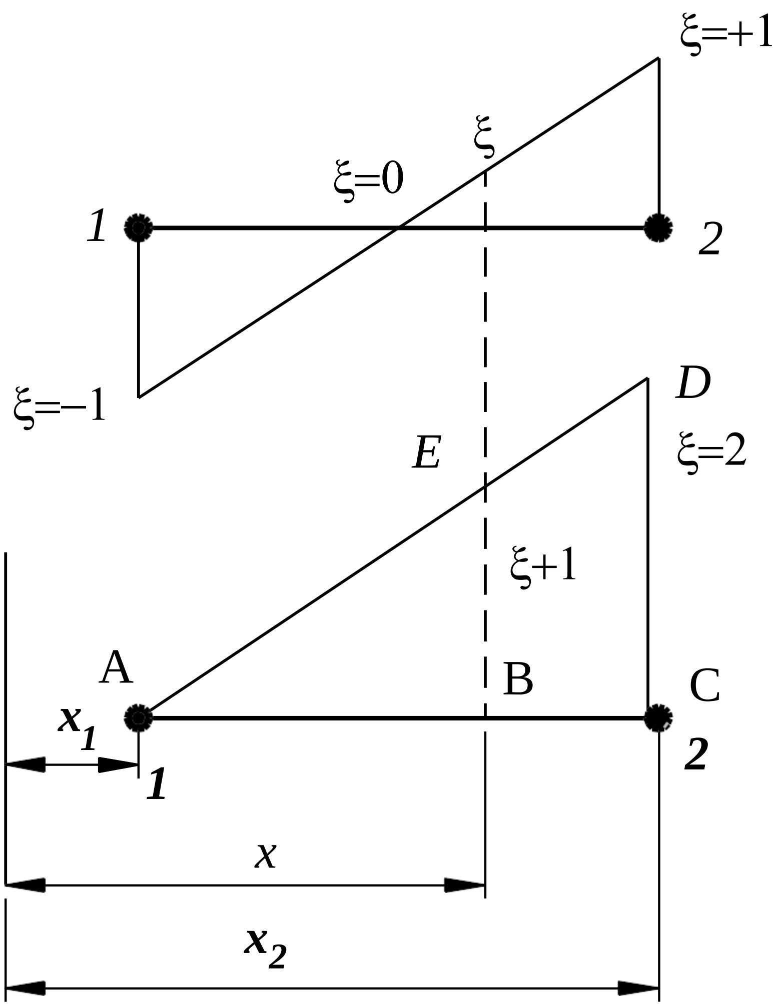


1-D Natural coordinate system



2-D Natural coordinate system

Relation between global & Natural Coordinate system



Consider an one dimensional bar element represented in natural coordinates as shown in fig.

Also the variation of natural coordinate is as shown in fig.

From similar triangles

ABE & ACD, $\dfrac{AB}{AC} = \dfrac{BE}{CD}$

$$\Rightarrow \frac{x - x_1}{x_2 - x_1} = \frac{\xi + 1}{2}$$

$$i.e.\ \xi + 1 = \frac{2\left(x - x_1\right)}{\left(x_2 - x_1\right)}$$

$$\therefore \xi = \frac{2\left(x - x_1\right)}{\left(x_2 - x_1\right)} - 1$$

Shape Functions

- Shape functions are defined as the interpolation functions used to interpolate the value of the field variable *(ex: displacement)* at any point within the element in terms of nodal values.

Mathematically, displacement at any point within the element is given by $u(x) = \sum_{i=1}^{n} N_i u_i$; where $'n'$ is the number of nodes

N_i are the shape functions & u_i are the nodal displacement components in x-direction. Thus, For a two noded element, $u(x) = N_1 u_1 + N_2 u_2$ where N_1 & N_2 are the shape functions & u_1 & u_2 are the displacements at node 1 & 2 respectively.

For a two dimensional model, displacement at any point is;

$$u(x, y) = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} N_i u_i \\ \sum_{i=1}^{n} N_i v_i \end{bmatrix}$$

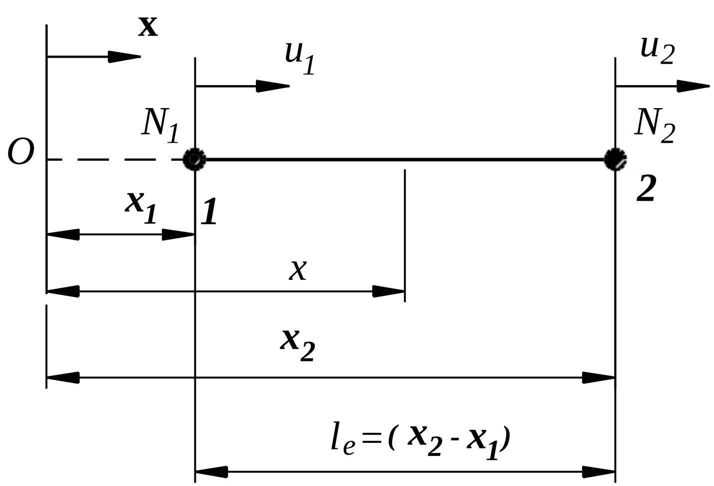
For a three noded triangular element,

$$u(x) = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v(x) = N_1 v_1 + N_2 v_2 + N_3 v_3$$

where N_1, N_2 & N_3 are the shape functions, u_1, u_2 & u_3 & v_1, v_2 & v_3 are the nodal displacements in *x and y* directions.

Shape Functions for 1 D bar element In terms of Cartesian coordinates



Consider a 1-D bar element of length l_e with a node at each end, & each node has one DOF.

The variation of displacement inside the element is given by $u = a_o + a_1 x$ where a_o & a_1 are the generalized coordinates to be found from $BC's$

$At\ x = x_1,\ u = u_1$ & $At\ x = x_2,\ u = u_2$

$\Rightarrow u_1 = a_o + a_1 x_1$ & $u_2 = a_o + a_1 x_2$

Thus, $(u_2 - u_1) = a_1 (x_2 - x_1)$

or $a_1 = \dfrac{(u_2 - u_1)}{(x_2 - x_1)}$

Substituting the value of a_1 into equation of u_1; $u_1 = a_o + \frac{(u_2 - u_1)}{(x_2 - x_1)} x_1$

$\therefore a_o = u_1 - \frac{(u_2 - u_1)}{(x_2 - x_1)} x_1 = \frac{(u_1 x_2 - u_2 x_1)}{(x_2 - x_1)}$ Substituting the values of a_0 & a_1

into equation of u, we get $u = \frac{(u_1 x_2 - u_2 x_1)}{(x_2 - x_1)} + \frac{(u_2 - u_1)}{(x_2 - x_1)} x$

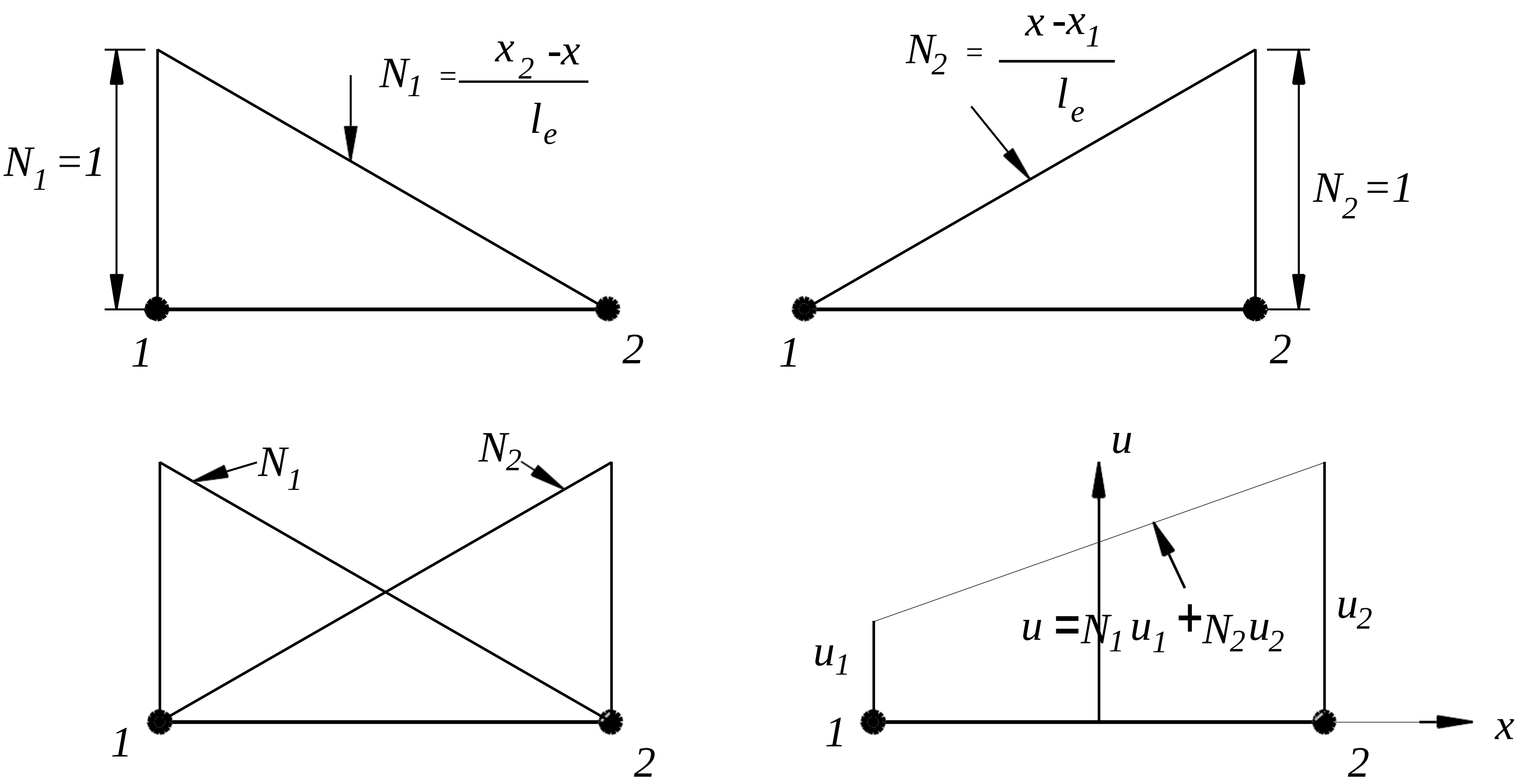
$u = \frac{(u_1 x_2 - u_2 x_1)}{l_e} + \frac{(u_2 - u_1)}{l_e} x$ where $l_e = (x_2 - x_1)$ is the length of the 1 D

bar element. Re-arranging the terms, $u = \frac{(u_1 x_2 - u_2 x_1) + u_2 x - u_1 x}{l_e}$

$u = \frac{(x_2 - x)}{l_e} u_1 + \frac{(x - x_1)}{l_e} u_2$ Also $u = N_1 u_1 + N_2 u_2$ Comparing the two equations;

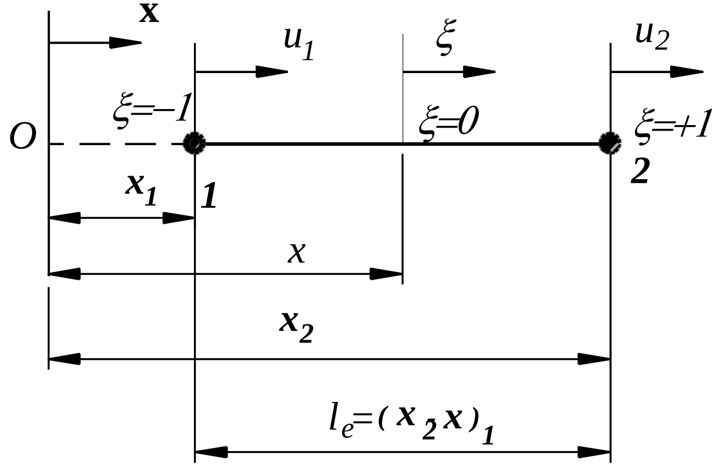
$N_1 = \frac{(x_2 - x)}{l_e}, N_2 = \frac{(x - x_1)}{l_e}$ Thus, values of shape functions at nodes 1 & 2 are

$$[N] = [N_1 \;\; N_2] = \left[\frac{(x_2 - x)}{l_e}, \frac{(x - x_1)}{l_e} \right]$$



Variation of shape function for 1 D bar element

Shape Functions for 1 D bar element In terms of Natural coordinates



Consider a 1-D bar element of length l_e with a node at each end, & each node has one DOF.

The variation of displacement inside the element is given by $u = a_o + a_1\xi$ where a_o & a_1 are the generalized coordinates to be found from $BC's$

At node 1; $\xi = -1, u = u_1$

At node 2,, $\xi = +1, u = u_2$

$\Rightarrow u_1 = a_o - a_1$ & $u_2 = a_o + a_1$

Thus, $a_o = \dfrac{(u_1 + u_2)}{2}$ & $a_1 = \dfrac{(u_2 - u_1)}{2}$

Substituting the values of a_o & a_1 into equation of u;

$$u = \frac{(u_1 + u_2)}{2} + \frac{(u_2 - u_1)}{2}\xi$$

Re-arranging the terms,

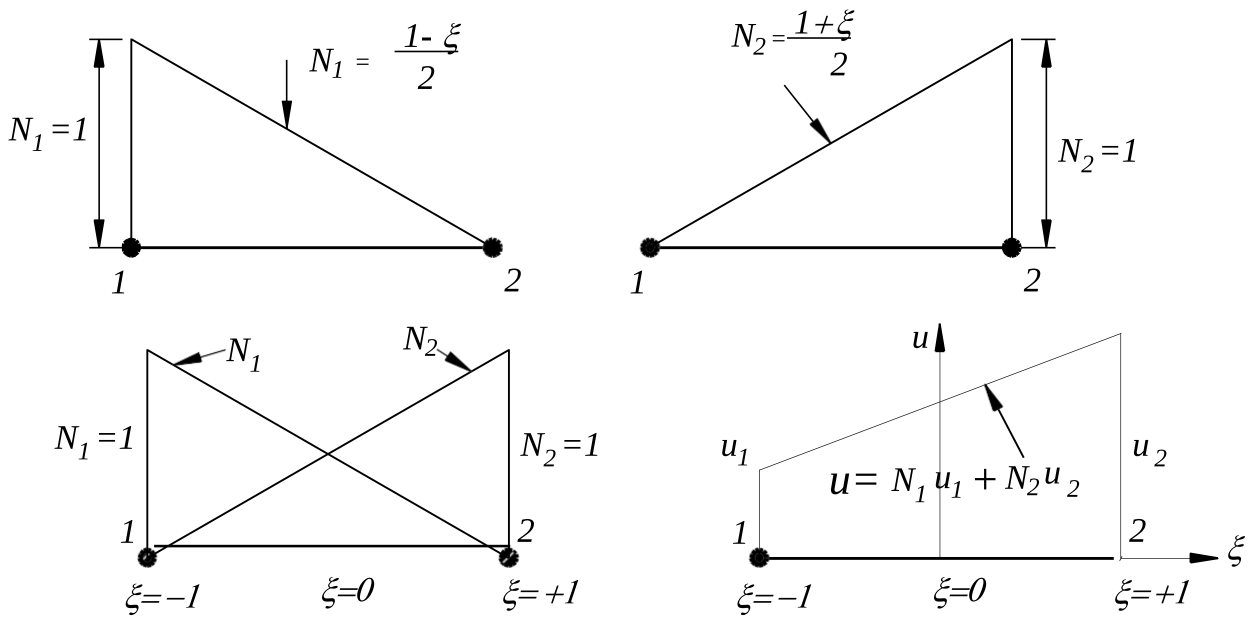
$$u = \frac{(1-\xi)}{2} u_1 + \frac{(1+\xi)}{2} u_2$$

Also $u = N_1 u_1 + N_2 u_2$, Comparing the two equations;

$$N_1 = \frac{(1-\xi)}{2}, \quad N_2 = \frac{(1+\xi)}{2}$$

Values of shape functions at nodes 1 & 2 are

$$[N] = \left[\frac{(1-\xi)}{2}, \frac{(1+\xi)}{2} \right]$$



Variation of shape function for 1 D bar element

Properties of Shape functions

1. The value of a shape function at a specified point is unity & at any other point its value is zero.

i.e. @ node 1, $N_1 = 1$, @ node 2, $N_1 = 0$ @ node 1, $N_2 = 0$, @ node 2, $N_2 = 1$

2.The sum of shape functions is unity.

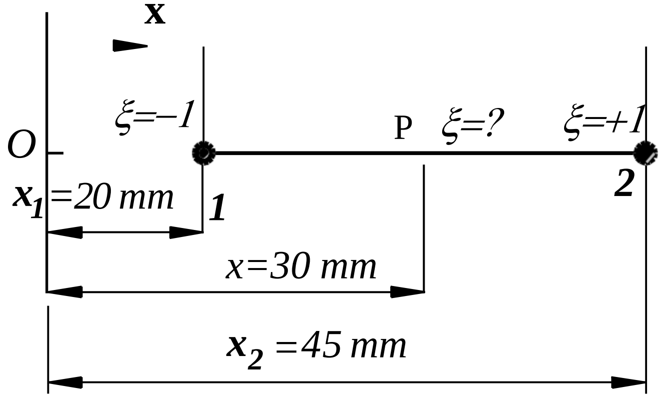
$$i.e.\ \mathrm{N}_1 = \left(\frac{1-\xi}{2}\right) \ \&\ \mathrm{N}_2 = \left(\frac{1+\xi}{2}\right), \Rightarrow N_1 + N_2 = 1$$

3.The derivative of shape function is constant.

$$i.e.\ \frac{dN_1}{d\xi} = -\frac{1}{2}, \quad \frac{dN_2}{d\xi} = +\frac{1}{2}$$

Q. Determine the value of ξ and shape functions N_1 & N_2 for a 1-D bar element as showwn in fig at point P, if;

$$u_1 = 0.003\ mm,\ u_2 = -0.005\ mm$$



Solution : *Natural coordinate ξ at point P is*

$$\xi_{@x=30} = \frac{2(x-x_1)}{(x_2-x_1)} - 1 = \frac{2(30-20)}{(45-20)} - 1 = \mathbf{-0.2}$$

$\therefore$ *Values of Shape functions at P are*

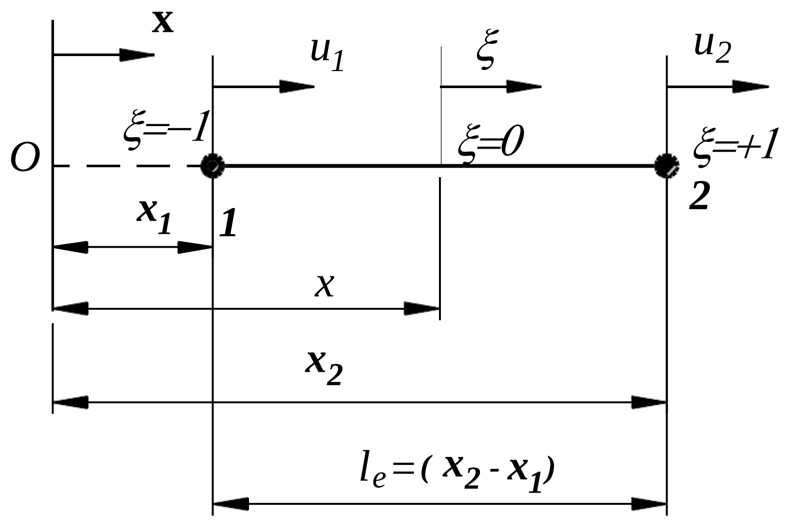
$$N_1 = \left(\frac{1-\xi}{2}\right) = \left(\frac{1-(-0.2)}{2}\right) = \mathbf{0.6}$$

$$N_2 = \left(\frac{1+\xi}{2}\right) = \left(\frac{1+(-0.2)}{2}\right) = \mathbf{0.4}$$

$\therefore$ *Displacement at* $P = u = N_1u_1 + N_2u_2$

$$\Rightarrow u = 0.6(0.003)+0.4(-0.005) = \mathbf{-2\times10^{-4}\ mm}$$

Derivation of strain matrix & strain-displacement [B] matrix



We know that strain in an element is given by $\varepsilon = \dfrac{\partial u}{dx}$

By parametric differentiation, $\varepsilon = \dfrac{\partial u}{d\xi} \times \dfrac{\partial \xi}{\partial x}$

The field variable $u = N_1 u_1 + N_2 u_2$ Where N_1 & N_2 are shape functions given by;

$$u = \left(\frac{1-\xi}{2}\right)u_1 + \left(\frac{1+\xi}{2}\right)u_2 \Rightarrow \frac{\partial u}{d\xi} = \frac{(u_2 - u_1)}{2}$$

Derivation of strain matrix & strain-displacement [B] matrix..

Also $\xi = \frac{2(x - x_1)}{l_e} - 1 = \frac{2(x - x_1)}{(x_2 - x_1)} - 1 \quad \therefore \frac{\partial \xi}{\partial x} = \frac{2}{(x_2 - x_1)} = \frac{2}{l_e}$

where l_e = length of element. Substituting for $\frac{\partial u}{\partial \xi}$ & $\frac{\partial \xi}{\partial x}$ in equation for ε,

$\varepsilon = \left(\frac{-u_1 + u_2}{2} \right) \times \frac{2}{l_e}$ In the matrix form, strain matrix $\varepsilon = \frac{1}{l_e} [-1 \;\; 1] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

i.e. Strain matrix $\varepsilon = [B]\{u\}$, where

$$[B] = \frac{1}{l_e} [-1 \;\; 1] \quad \cdots\cdots (i)$$ ***is the strain -displacement matrix.***

From Hooke's law, stress $\sigma = E\varepsilon \Rightarrow \boldsymbol{\sigma = E[B]\{u\}} \quad \cdots\cdots$ ***(ii)***

Eqn (ii) is the ***stress matrix*** *for 1 - D bar element.*

Derivation of stiffness matrix using strain-displacement matrix

Strain energy for an element is given by $SE = \frac{1}{2}\int_V \sigma^T \varepsilon \, dV$

For 1-D bar element, $Volume = c/s\ area\ (A) \times length\ of\ element\ l_e$

$$\left[\therefore \text{Intergral over volume } = \text{Area} \times \text{Integral over length } \int_V dV = \int_{l_e} A.dx \right]$$

Also, $\varepsilon = [B]\{u\}$ & $\sigma = E[B]\{u\}$ Substituting,

$$SE = \frac{1}{2}\int_{l_e} \left(E[B]\{u\}\right)^T [B]\{u\} A.dx$$

As E is a constant term, & $\left([B]\{u\}\right)^T = \{u\}^T [B]^T$, Strain energy becomes;

$$SE = \frac{1}{2}\{u\}^T \int_{l_e} \left([B]^T E[B] A.dx\right)\{u\} = \frac{1}{2}\{u\}^T [k_e]\{u\}$$ where $\boldsymbol{k_e}$ ***is elemental stiffness matrix*** given by $[\boldsymbol{k_e}] = \int \left(\boldsymbol{[B]^T E[B] A.dx}\right)$

Derivation of stiffness matrix ....

$$[k_e] = \int_{l_e} \left([B]^T E[B]A.dx\right) \text{ Substituting } dx = \frac{l_e}{2} d\xi$$

$$[k_e] = [B]^T E[B]A.\frac{l_e}{2}\int_{-1}^{+1} d\xi = AEl_e[B]^T[B]\xi \Big|_{-1}^{+1} = AEl_e[B]^T[B]$$

$$\text{Also } [B]^T[B] = \frac{1}{l_e}\begin{bmatrix}-1\\1\end{bmatrix} \times \frac{1}{l_e}\begin{bmatrix}-1 & 1\end{bmatrix} = \frac{1}{l_e^2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

$$\therefore [k_e] = \frac{AE}{l_e}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} \cdots\cdots \text{(iii) is the Elemental stiffness matrix.}$$

Derivation of Load Vector

(i) Load vector due to body force :

Work potential due to body force is given by $\int_V u^T f \; dV = \int_{l_e} u^T f \; A\, dx$

But $u = [N]\{u\}$ and $dx = \frac{l_e}{2} d\xi \Rightarrow WP_{\text{Body force}} = \int_{l_e} \left([N]\{u\}\right)^T f A \,\frac{l_e}{2} d\xi$

$$Also \left([N]\{u\}\right)^T = \{u\}^T [N]^T$$

$$WP_{\text{Body force}} = \frac{fAl_e}{2}\{u\}^T \int_{-1}^{+1} [N]^T d\xi = \frac{fAl_e}{2}\{u\}^T \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\because \int_{-1}^{+1} [N]^T d\xi = \begin{bmatrix} \int_{-1}^{+1} \left(\frac{1-\xi}{2}\right) d\xi \\ \int_{-1}^{+1} \left(\frac{1+\xi}{2}\right) d\xi \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\xi - \frac{\xi^2}{2}\right)_{-1}^{+1} \\ \frac{1}{2}\left(\xi + \frac{\xi^2}{2}\right)_{-1}^{+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(2-0) \\ \frac{1}{2}(2+0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore WP_{\text{body force}} = \{u\}^T \{f\} \ldots$ ***(iv) where*** $\{f\} = \frac{Al_e f}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ is the body force vector

Derivation of Load Vector...

(ii) Load vector due to surface traction force :

Work potential due to surface traction is given by $\int_s u^T T \, ds$

For1-D bar element, traction is considered per unit length of surface.

$$\therefore \mathrm{WP}_{Traction} = \int_{l_e} u^T T \, dx, \quad \text{But } u = [N]\{u\} \text{ and } dx = \frac{l_e}{2} d\xi$$

$$\Rightarrow WP_{Traction} = \int_{l_e} \left([N]\{u\}\right)^T T \, \frac{l_e}{2} d\xi \quad Also \left([N]\{u\}\right)^T = \{u\}^T [N]^T$$

$$WP_{Traction} = \frac{Tl_e}{2} \{u\}^T \int_{-1}^{+1} [N]^T d\xi = \frac{Tl_e}{2} \{u\}^T \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\therefore \boldsymbol{WP_{Traction}} = \{u\}^T \{T\} \ldots (v) \quad \textbf{\textit{where}} \ \{T\} = \frac{Tl_e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$ is the surface traction vector

Potential Energy functional for a continuum

PE functional for an element $= \left(SE - WP\right)$

$$= \frac{1}{2}\{u\}^T [k_e]\{u\} - \{u\}^T \{f\} - \{u\}^T \{T\} - \{u\}^T P_i$$

For the whole continuum, PE functional may be written as;

$$\Pi = \frac{1}{2}\{U\}^T [K]\{U\} - U^T F \quad \text{where;}$$

U is the global displacement vector K is the global stiffness matrix

F is global force vector (Body force+Traction+Point loads)

$$\Rightarrow F = \left[\frac{fA_e l_e}{2}\begin{Bmatrix}1\\1\end{Bmatrix} + \frac{Tl_e}{2}\begin{Bmatrix}1\\1\end{Bmatrix} + P_i \right]$$

Properties of Stiffness matrix

1. *The stiffness matrix is a banded & symmetric matrix*
2. *If there are'n' number of nodes with one degree of freedom each, then order of stiffness matrix is* $n \times n$.
3. *The main diagonal elements of the stiffness matrix are always positive.*
4. *If rigid body motion is not prevented by sufficient boundary conditions the stiffness matrix becomes singular. (i.e. its determinant becomes zero)*

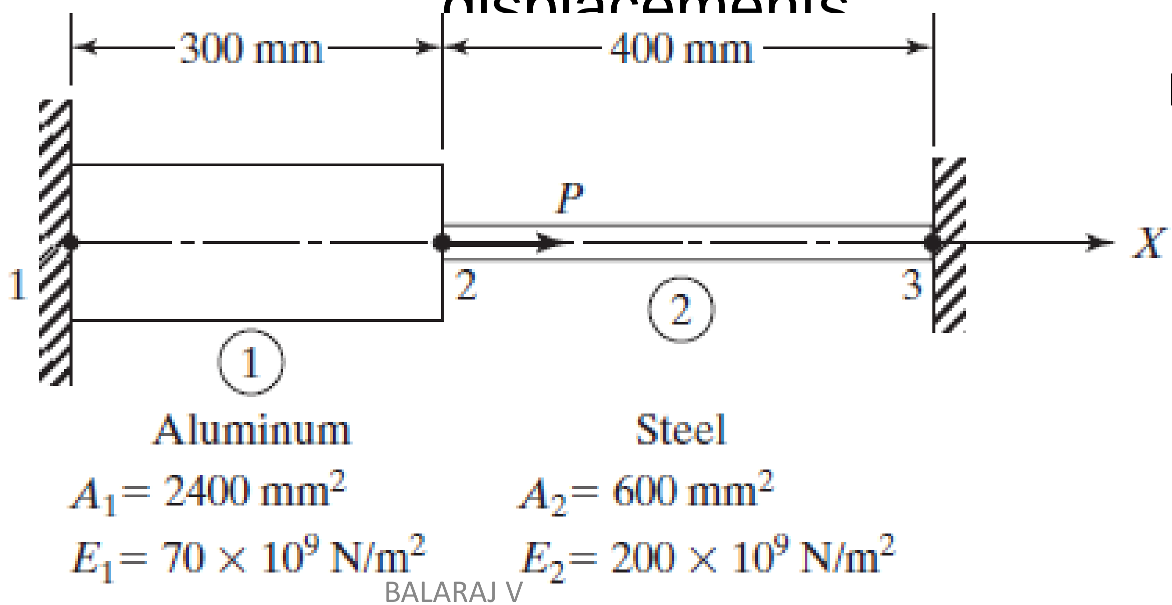
Problem 1

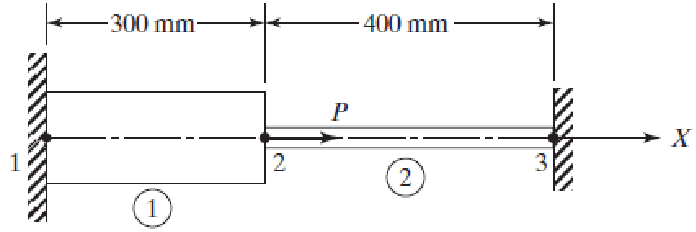
Consider the bar shown in Fig. An axial load P = 200 KN is applied as shown.

Using elimination approach for handling boundary conditions, (a)Determine the nodal displacements

each

forces.





***Stiffness matrices*:**

$$[k]^{(1)} = \frac{2400 \times 70 \times 10^3}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{bmatrix} \overset{1}{0.56} & \overset{2}{-0.56} \\ -0.56 & 0.56 \end{bmatrix}\begin{matrix} 1 \\ 2 \end{matrix} = \begin{bmatrix} \overset{1}{0.56} & \overset{2}{-0.56} & \overset{3}{0} \\ -0.56 & 0.56 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

$$[k]^{(2)} = \frac{600 \times 200 \times 10^3}{400}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{bmatrix} \overset{2}{0.3} & \overset{3}{-0.3} \\ -0.3 & 0.3 \end{bmatrix}\begin{matrix} 2 \\ 3 \end{matrix} = \begin{bmatrix} \overset{1}{0} & \overset{2}{0} & \overset{3}{0} \\ 0 & 0.3 & -0.3 \\ 0 & -0.3 & 0.3 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

***Global stiffness matrix*:** $[K] = [k]^{(1)} + [k]^{(2)}$

$$[K] = 10^6 \begin{bmatrix} \overset{1}{0.56} & \overset{2}{-0.56} & \overset{3}{0} \\ -0.56 & 0.86 & -0.3 \\ 0 & -0.3 & 0.3 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

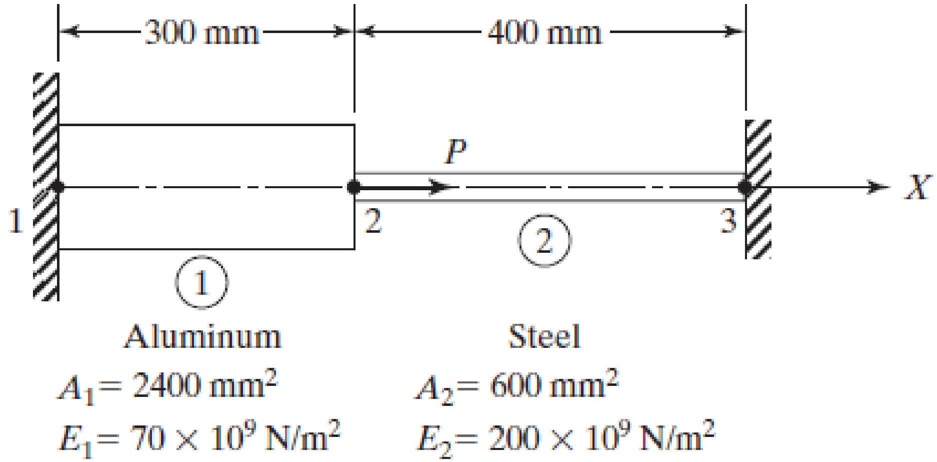
Global load vector :

$$\text{Global load vector } \{F\} = \begin{Bmatrix} 0 \\ 200\times10^3 \\ 0 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

Equilibrium Equation : $[K]\{U\} = \{F\}$ Using fixed bc's at nodes 1 & 3,

$$\Rightarrow 10^6 \begin{bmatrix} \overset{1}{0.56} & \overset{2}{-0.56} & \overset{3}{0} \\ -0.56 & 0.86 & -0.3 \\ 0 & -0.3 & 0.3 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 200\times10^3 \\ 0 \end{Bmatrix} \quad \therefore 0.86\times10^6 u_2 = 200\times10^3$$

$u_2 = 0.2326$ ***mm***, $u_1 = u_3 = 0$



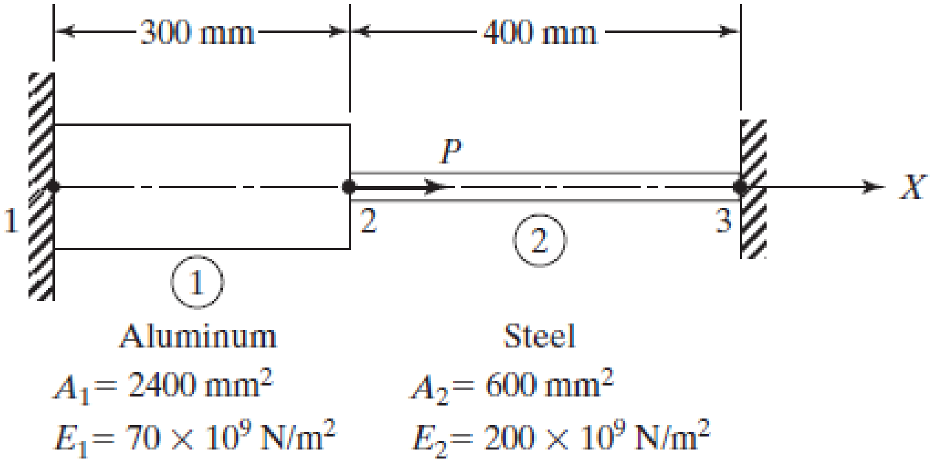
Stresses & strains :

$$\varepsilon^{(1)} = \frac{u_2 - u_1}{L_1} = \frac{0.2326 - 0}{300} = \mathbf{7.752 \times 10^{-4}}$$

$$\sigma^{(1)} = E\varepsilon^{(1)} = 70 \times 10^3 \times 7.752 \times 10^{-4} = \mathbf{54.26\ N/mm^2}$$

$$\varepsilon^{(2)} = \frac{u_3 - u_2}{L_2} = \frac{0 - 0.2326}{400} = \mathbf{-5.815 \times 10^{-4}}$$

$$\sigma^{(2)} = E\varepsilon^{(2)} = 200 \times 10^3 \times (-5.815 \times 10^{-4}) = \mathbf{-116.3\ N/mm^2}$$



Reactions at Nodes: $\{R\} = [K]\{U\} - F$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 10^6 \begin{bmatrix} 0.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.3 \\ 0 & -0.3 & 0.3 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 200\times10^3 \\ 0 \end{Bmatrix}$$

(Matrix columns labelled 1, 2, 3; rows labelled 1, 2, 3.)

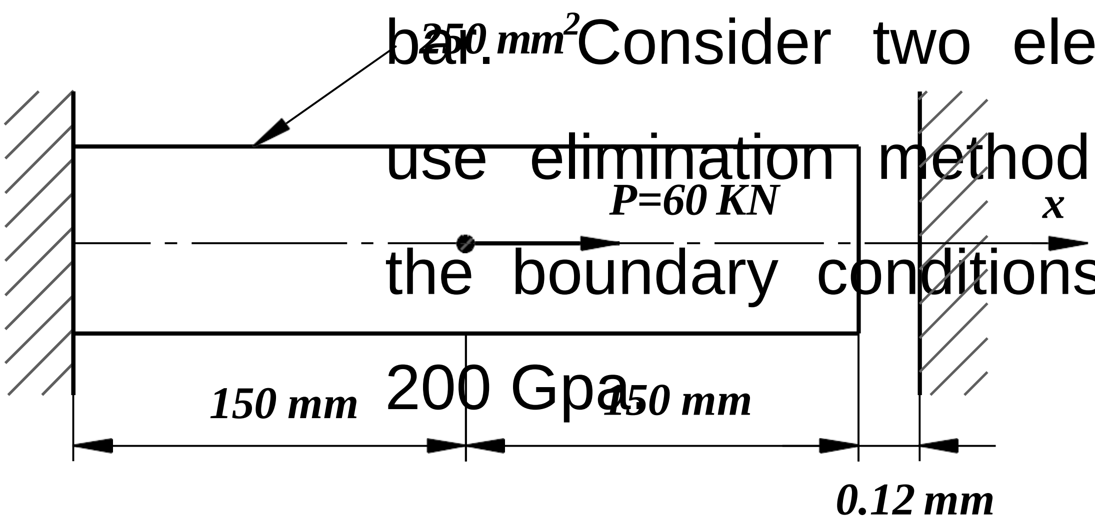
$R_1 = 10^6(-0.56\times0.2326) - 0 =$ ***-130.26 KN***

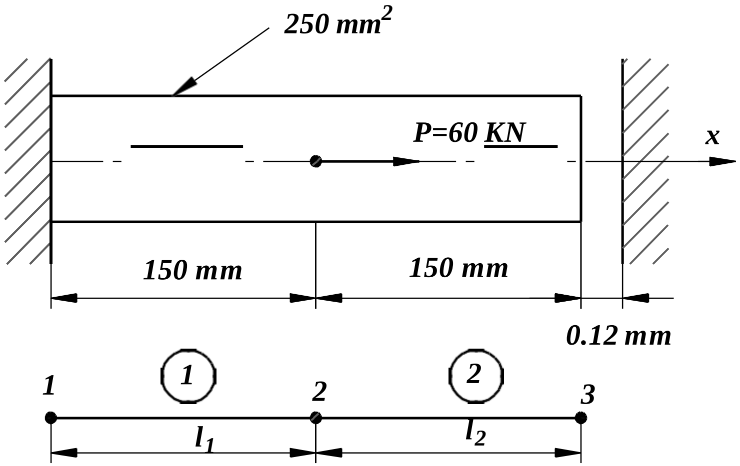
$R_2 = 10^6(0.86\times0.2326) - 200\times10^3 =$ ***0***

$R_3 = 10^6(-0.3\times0.2326) - 0 =$ ***-69.78 KN***

Problem 2

A bar having uniform cross sectional area of 250 mm^2 is subjected to a load P= 60 KN as shown in fig. Determine the displacement field, stress & support reactions in the bar. Consider two elements and use elimination method to handle the boundary conditions. Take E= 200 Gpa.





Stiffness matrix of an element is

$$[k]^{(e)} = \frac{A_e E_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Element Stiffness matrices :

$$[k]^{(1)} = \frac{250\times 200\times 10^3}{150}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{10^6}{3}\begin{bmatrix} \overset{1}{1} & \overset{2}{-1} \\ -1 & 1 \end{bmatrix}\begin{matrix}1\\2\end{matrix} = \frac{10^6}{3}\begin{bmatrix} \overset{1}{1} & \overset{2}{-1} & \overset{3}{0} \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix}1\\2\\3\end{matrix}$$

$$[k]^{(2)} = \frac{250\times 200\times 10^3}{150}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{10^6}{3}\begin{bmatrix} \overset{2}{1} & \overset{3}{-1} \\ -1 & 1 \end{bmatrix}\begin{matrix}2\\3\end{matrix} = \begin{bmatrix} \overset{1}{0} & \overset{2}{0} & \overset{3}{0} \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{matrix}1\\2\\3\end{matrix}$$

Global stiffness matrix : $[K] = [k]^{(1)} + [k]^{(2)}$

$$[K] = \frac{10^6}{3}\begin{bmatrix} \overset{1}{1} & \overset{2}{-1} & \overset{3}{0} \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

Global displacement vector : $[U] = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ u_2 \\ 0.12 \end{Bmatrix}$

Global load vector : $[F] = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 60 \times 10^3 \\ 0 \end{Bmatrix}$

Equilibrium Equation: $[K]\{U\}=\{F\}$

Elimination Method of applying boundary conditions :

Using bc's at nodes 1 & 3, as node 1 is fixed, the corresponding row & column may be eliminated. But at node 3, a specified displacement $a_3 = 0.12$ mm is given.

Hence the force vector must be modified as; $\begin{Bmatrix} F_1 - k_{13}a_3 \\ F_2 - k_{23}a_3 \\ F_3 - k_{33}a_3 \end{Bmatrix}$

Now, first row & first column & third row & third column may be eliminated.

$$\Rightarrow \frac{10^6}{3}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0.12 \end{Bmatrix} = \begin{Bmatrix} F_1 - k_{13}a_3 \\ F_2 - k_{23}a_3 \\ F_3 - k_{33}a_3 \end{Bmatrix} = \begin{Bmatrix} 0-0 \\ 60\times10^{-3} - \left(\frac{10^6}{3}\right)(-1)\,0.12 \\ 0-\left(\frac{10^6}{3}\right)0.12 \end{Bmatrix}$$

$$\therefore \frac{10^6}{3}(2u_2) = 60\times10^3 - \left(-\frac{10^6}{3}\times0.12\right) \Rightarrow$$ ***u_2 = 0.15 mm. Also u_1 = 0, u_3 = 0.12 mm***

Stresses & strains :

$$\varepsilon^{(1)} = \frac{u_2 - u_1}{L_1} = \frac{0.15 - 0}{150} = \mathbf{1 \times 10^{-3}},$$

$$\sigma^{(1)} = E\varepsilon^{(1)} = 200 \times 10^3 \times 1 \times 10^{-3} = \mathbf{200\ N/mm^2}$$

$$\varepsilon^{(2)} = \frac{u_3 - u_2}{L_2} = \frac{0.12 - 0.15}{150} = \mathbf{-2 \times 10^{-4}},$$

$$\sigma^{(2)} = E\varepsilon^{(2)} = 200 \times 10^3 \times (-2 \times 10^{-4}) = \mathbf{-40\ N/mm^2}$$

Reactions at fixed supports : $\{R\} = [K]\{U\} - F$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \frac{10^6}{3} \begin{bmatrix} \overset{1}{1} & \overset{2}{-1} & \overset{3}{0} \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{Bmatrix} 0 \\ 0.15 \\ 0.12 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} - \begin{Bmatrix} 0 \\ 60 \times 10^3 \\ 0 \end{Bmatrix}$$

$$R_1 = \frac{10^6}{3}(-1 \times 0.15) - 0 = \mathbf{-50\ KN},$$

$$R_3 = \frac{10^6}{3}(-1 \times 0.15 + 1 \times 0.12) - 0 = \mathbf{-10\ KN}$$

$$\textbf{\textit{Equilibrium Equation}}: [K]\{U\} = \{F\}$$

Penalty approach of applying boundary conditions :

In this approach, the fixed nodes may be modelled as those having

a very high stiffness C, where $C = \left| Max\ K_{ij} \right| \times 10^4$

$$Here,\ C = \frac{10^6}{3} \times 2 \times 10^4 = \frac{10^6}{3}(20000) \quad (\text{i.e. } C = 0.667 \times 10^{10})$$

Add this value to stiffness terms at node 1 & 3. Also add $Ca_1 \& Ca_3$

$$Hence,\ k_{11} = k_{33} = \frac{10^6}{3}(20000 + 1) = \frac{10^6}{3}(20001)$$

$$F_1 = Ca_1 = (0.667 \times 10^{10} \times 0) = 0$$

$$F_3 = Ca_3 = (0.667 \times 10^{10} \times 0.12) = 800 \times 10^6$$

$$\Rightarrow \frac{10^6}{3}\begin{bmatrix} \overset{1}{20001} & \overset{2}{-1} & \overset{3}{0} \\ -1 & 2 & -1 \\ 0 & -1 & 20001 \end{bmatrix}\begin{matrix}1\\2\\3\end{matrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}\begin{matrix}1\\2\\3\end{matrix} = \begin{Bmatrix} 0 \\ 60\times10^3 \\ 800\times10^6 \end{Bmatrix}$$

$$\therefore \frac{10^6}{3}\left[(20001u_1) - u_2\right] = 0 \Rightarrow \boldsymbol{20001u_1 - u_2 = 0 \cdots (i)}$$

$$\frac{10^6}{3}\left[(-u_1) + 2u_2 - u_3\right] = 60\times10^3 \Rightarrow \boldsymbol{-u_1 + 2u_2 - u_3 = 0.18 \cdots (ii)}$$

$$\frac{10^6}{3}\left[(0 - u_2 + 20001u_3\right] = 800\times10^6 \Rightarrow \boldsymbol{-u_2 + 20001u_3 = 2400 \cdots (ii)}$$

Solving, $\boldsymbol{u_1 = 7.4998\times10^{-6}}$ ***mm,*** $\boldsymbol{u_2 = 0.15}$***mm,*** $\boldsymbol{u_3 = 0.1200015}$ ***mm,***

Reactions at fixed supports : $R = -C(q_i - a_i)$

$$R_1 = -C(q_1 - a_1) = -0.6667\times10^{10}(7.4998\times10^{-6} - 0) = \boldsymbol{-50\ KN}$$

$$R_3 = -C(q_3 - a_3) = -0.6667\times10^{10}(0.1200015 - 0.12) = \boldsymbol{-10\ KN}$$

Note : In penalty method, do not round off the displacements to second or third decimal place. Keep all the digits after decimal.

Temperature Effects:

If there is a change in temperature ΔT of a 1 D bar element, the load vector may be modified as; $F = \sum_e f^e + T^e + \theta^e + P$ where;

f^e = Body force, T^e = Traction force, P = Point load

θ^e is the additional load due to thermal effect, given by

$$\theta^e = \left(E \times A \times \alpha \times \Delta T \right)$$

where α = Coefficient of thermal expansion

A = Area of the element, E = Modulus of elasticity

Strain in the element is $\varepsilon = [B]\{u\} - \alpha \Delta T$

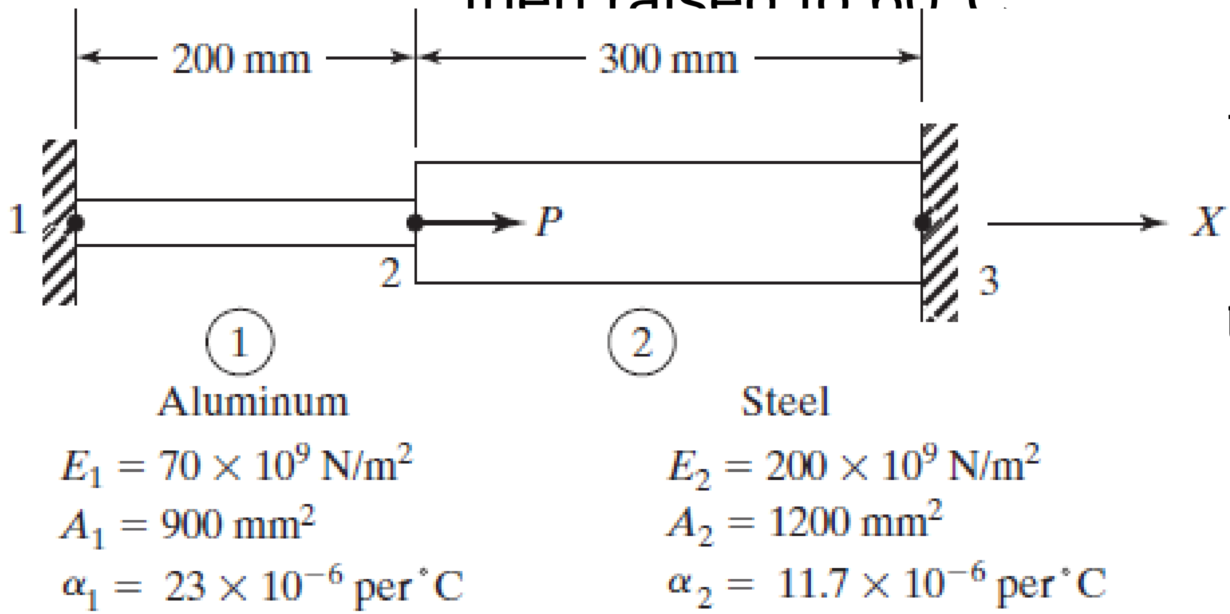
Stress in the element is $\sigma = E\varepsilon = \mathrm{E} \times \left([B]\{u\} - \alpha \Delta T \right)$

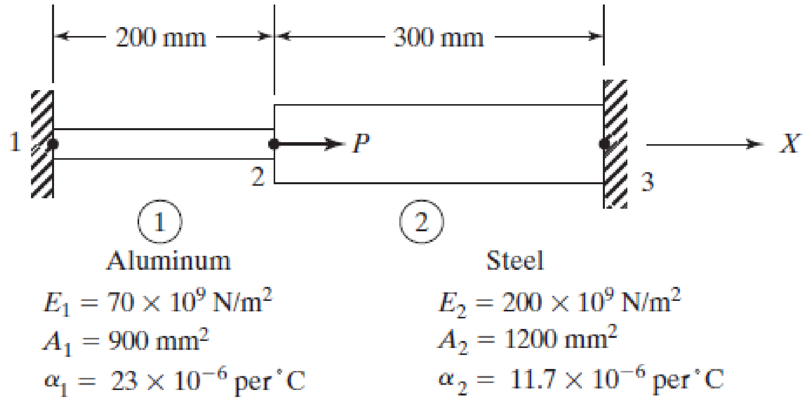
Problem 3

An axial load P = 300 KN is applied at 20°C to the rod as shown in Fig. The temperature is then raised to 60°C.

…atrices.

…t stresses.





Element Stiffness matrices :

Stiffness matrix of an element is $[k]^{(e)} = \frac{A_e E_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$[k]^{(1)} = \frac{900 \times 70 \times 10^3}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^3 \begin{bmatrix} \overset{1}{315} & \overset{2}{-315} \\ -315 & 315 \end{bmatrix}\begin{matrix} 1 \\ 2 \end{matrix} = 10^3 \begin{bmatrix} \overset{1}{315} & \overset{2}{-315} & \overset{3}{0} \\ -315 & 315 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

$$[k]^{(2)} = \frac{1200 \times 200 \times 10^3}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^3 \begin{bmatrix} \overset{2}{800} & \overset{3}{-800} \\ -800 & 800 \end{bmatrix}\begin{matrix} 2 \\ 3 \end{matrix} = 10^3 \begin{bmatrix} \overset{1}{0} & \overset{2}{0} & \overset{3}{0} \\ 0 & 800 & -800 \\ 0 & -800 & 800 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

Global Stiffness matrices : $[K] = [k]^{(1)} + [k]^{(2)}$

$$[K] = 10^3 \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 315 & -315 & 0 \\ -315 & 315 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} + 10^3 \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 800 & -800 \\ 0 & -800 & 800 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$\Rightarrow [K] = 10^3 \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 315 & -315 & 0 \\ -315 & 1115 & -800 \\ & 0 & 800 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

-80

0

Element Load Vectors:

Here there is a temperature change of $\Delta T = (60 - 20) = 40^0 C$

Load in element 1 due to ΔT is $\theta^{(1)} = (E_1 \times A_1 \times \alpha_1 \times \Delta T)\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$

$$\Rightarrow \theta^{(1)} = (70 \times 10^3 \times 900 \times 23 \times 10^{-6} \times 40)\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = 57.96 \times 10^3 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Similarly, Load in element 1 due to ΔT, $\theta^{(2)} = (E_2 \times A_2 \times \alpha_2 \times \Delta T)\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$

$$\Rightarrow \theta^{(2)} = (200 \times 10^3 \times 1200 \times 11.7 \times 10^{-6} \times 40)\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = 112.32 \times 10^3 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Also, there is point load at node 2 which is equal to $300 \times 10^3 N$.

Global Load Vector :

$$\{F\} = 10^3 \begin{Bmatrix} -57.96 \\ 57.96 - 112.32 + 300 \\ 112.32 \end{Bmatrix} = 10^3 \begin{Bmatrix} -57.96 \\ 245.64 \\ 112.32 \end{Bmatrix}$$

Equilibrium Equation : $[K]\{U\} = \{F\}$

$$\Rightarrow 10^3 \begin{bmatrix} \overset{1}{315} & \overset{2}{-315} & \overset{3}{0} \\ -315 & 1115 & -800 \\ 0 & -800 & 800 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 10^3 \begin{Bmatrix} -57.96 \\ 245.64 \\ 112.32 \end{Bmatrix}$$

Using fixed bc's at nodes 1 & 3, $u_1 = u_3 = 0$ Hence eliminating row & column numbers 1 & 3

$\therefore 10^3 \times 1115 \times u_2 = 10^3 \times 245.64 \;\therefore$ ***u_2 = 0.22 mm, u_1 = u_3 = 0***

Strains & stresses :

$$\varepsilon^{(1)} = \left(\frac{u_2 - u_1}{L_1}\right) - \alpha_1 \Delta T = \left(\frac{0.22 - 0}{200}\right) - \left(23\times10^{-6}\times 40\right) = \mathbf{1.8\times10^{-4}}$$

$$\sigma^{(1)} = E_1\,\varepsilon^{(1)} = 70\times10^{3}\times 1.8\times10^{-4} = \mathbf{12.6\ N/mm^2}$$

$$\varepsilon^{(2)} = \left(\frac{u_3 - u_2}{L_2}\right) - \alpha_2 \Delta T = \left(\frac{0 - 0.22}{300}\right) - \left(11.7\times10^{-6}\times 40\right) = \mathbf{-1.201\times10^{-3}}$$

$$\sigma^{(2)} = E\varepsilon^{(2)} = 200\times10^{3}\times(-1.201\times10^{-3}) = \mathbf{-240.2\ N/mm^2}$$

Reactions at fixed supports : $\{R\} = [K]\{U\} - F$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 10^3 \begin{bmatrix} \overset{1}{315} & \overset{2}{-315} & \overset{3}{0} \\ -315 & 1115 & -800 \\ 0 & -800 & 800 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{Bmatrix} 0 \\ 0.22 \\ 0 \end{Bmatrix} - 10^3 \begin{Bmatrix} -57.96 \\ 245.64 \\ 112.32 \end{Bmatrix}$$

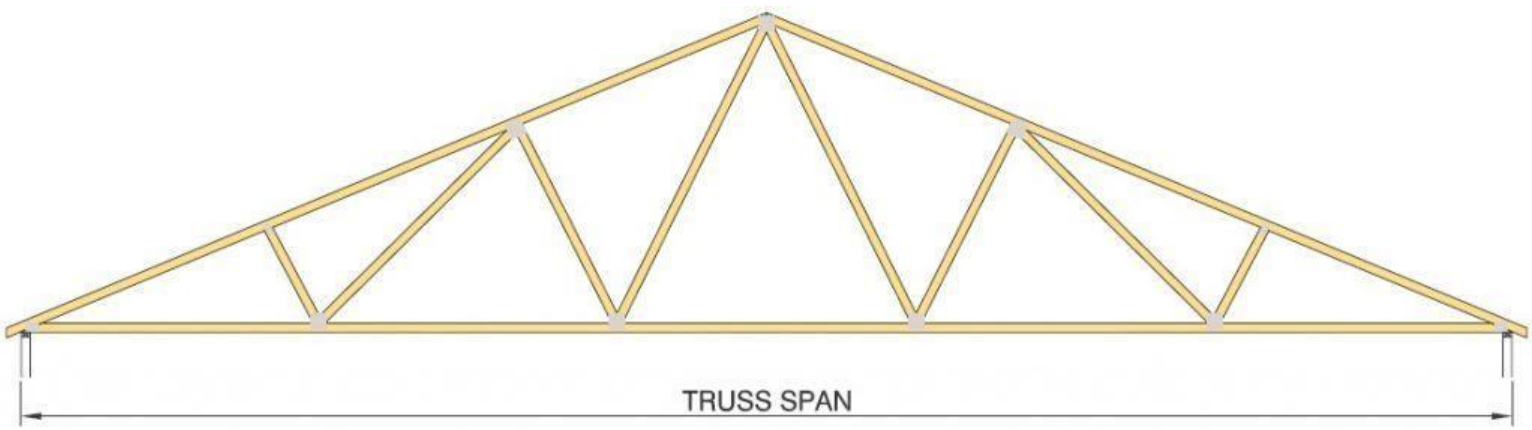
$$R_1 = 10^3\left[(-315\times0.22) - (-57.96)\right] = \mathbf{-11.34\ KN}$$

$$R_1 = 10^3\left[(1115\times0.22) - (245.64)\right] = \mathbf{-0.34\ KN}$$

$$R_3 = 10^3(-800\times0.22 - 112.32) = \mathbf{288.32 KN}$$

ANALYSIS OF TRUSSES

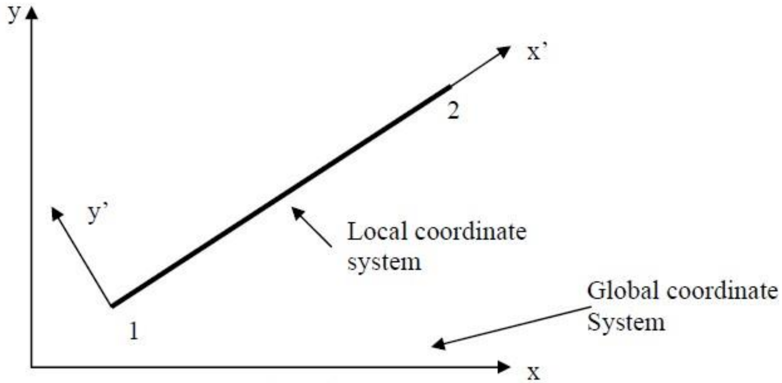
- A framework composed of members joined at their ends to form a structure is called a truss.
- Truss is used for supporting moving or stationary load. Bridges, roof supports, derricks, and other such structures are common example of trusses.
- When the members of the truss lie essentially in a single plane, the truss is called a plane truss .



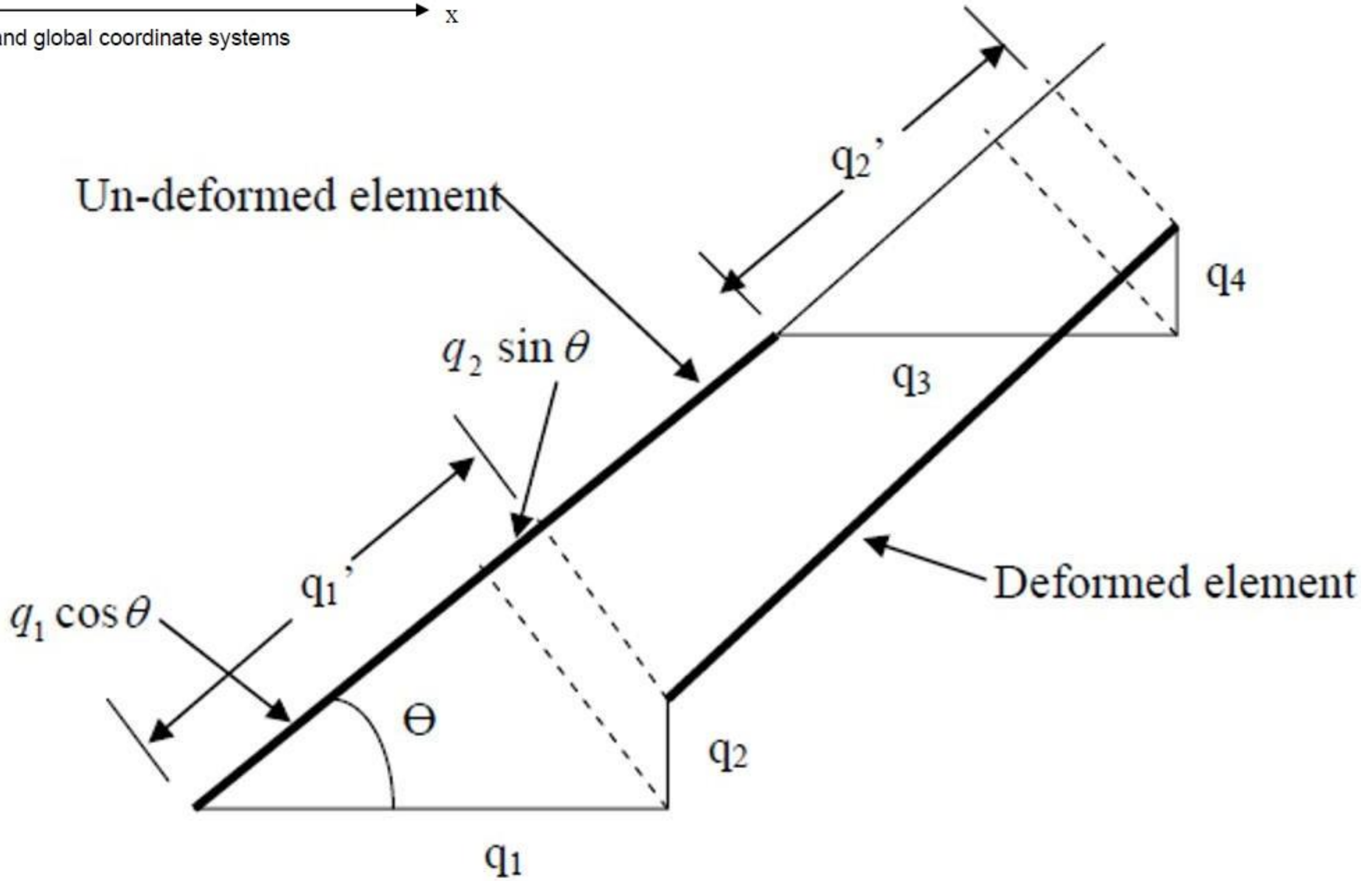
Common assumptions made in analysis of trusses

- It should be a prismatic member of a homogenous & isotropic material resisting a constant load.
- A load on a truss can only be applied at the joints (nodes).
- Due to the load applied each bar of a truss is either induced with tensile/compressive forces.
- The joints in a truss are assumed to be frictionless pin joints
- Self-weight of the bars are neglected.

Element stiffness matrix of Trusses

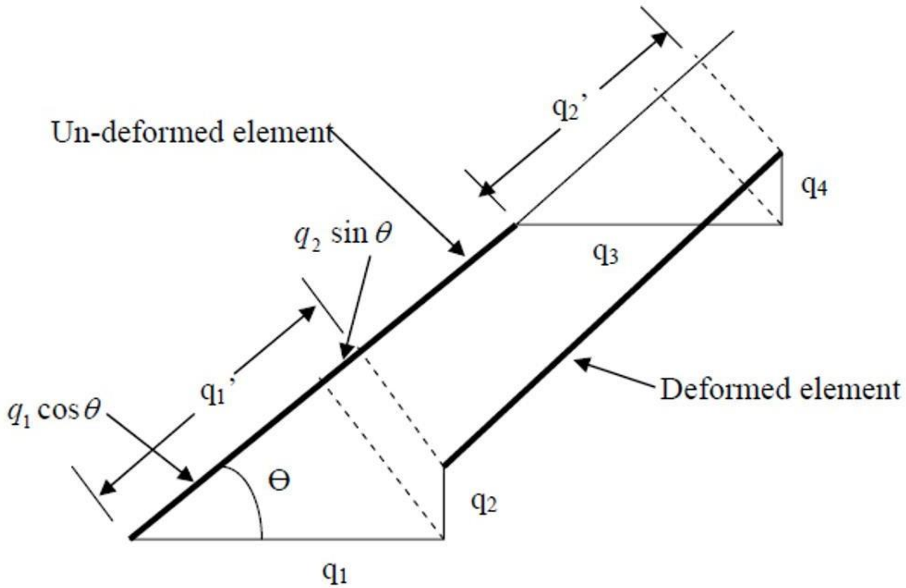


- Local and global coordinate systems



The deformation of an element in both local and global coordinate systems.

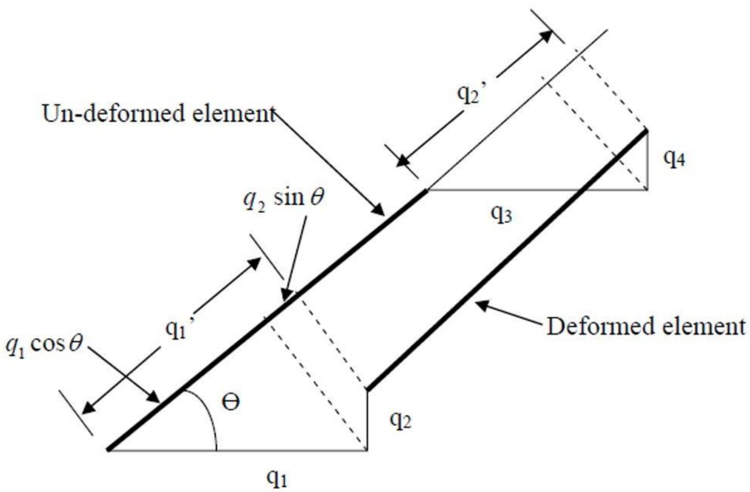
Element stiffness matrix of Trusses



The deformation of an element in both local and global coordinate systems.

- Fig shows a typical truss element in local & global coordinate system.
•Local coordinates vary with the orientation of the element where as the global coordinates remain fixed and does not depend on the orientation of the element.
•Let x & y be the global coordinates and each node has two dof.
•Let q_1 and q_2 be the x & y displacements at node 1 and q_3 and q_4 be the values at node 2.
•Similarly, q_1", q_2", q_3" and q_4" be the corresponding local displacements.

Element stiffness matrix of Trusses



The deformation of an element in both local and global coordinate systems.

From the fig, relationship between q and q' is ;

$$q_1^{'} = q_1 \cos\theta + q_2 \sin\theta$$

$$q_2^{'} = q_3 \cos\theta + q_4 \sin\theta$$

Let $l = \cos\theta$ and $m = \sin\theta$ be the direction cosines. Then

$$q_1^{'} = lq_1 + mq_2$$

$$q_2^{'} = lq_3 + mq_4$$

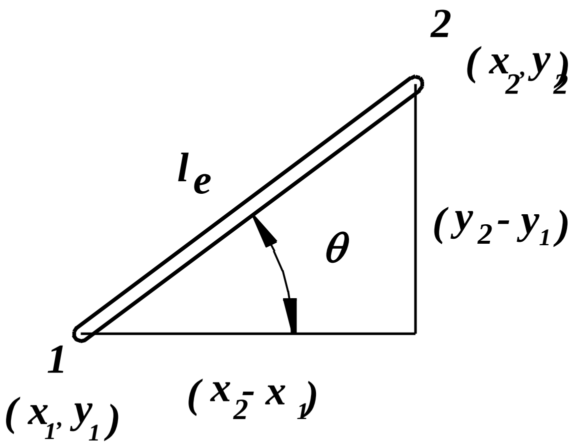
In the matrix form ;

$$\begin{Bmatrix} q_1^{'} \\ q_2^{'} \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = [L]\{q\}$$

Element stiffness matrix of Trusses

where $[L] = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$ is the ***transformation matrix.***

To find the direction cosines :



Direction Cosines

Let the coordinates of the ends of truss element whose length is l_e be as shown. From the fig, direction cosines are given by ;

$$l = \cos\theta = \frac{(x_2 - x_1)}{l_e}$$

$$m = \sin\theta = \frac{(y_2 - y_1)}{l_e}$$

The length of the element is

$$l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The truss element is equivalent to one dimensional bar element in local coordinates. Hence the element stiffness matrix is given

by; $k_e^{'} = \frac{A_e E_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ with usual notations.

(The single prime (') denotes local coordinate system)

The elemental strain energy for a truss element in local coordinate system is given by $U_e = \frac{1}{2} q'^T k' q'$

Stiffness matrix needs to be in global coordinate sytem.

Using $q' = Lq,\ U_e = \frac{1}{2}[Lq]^T k'[Lq] = \frac{1}{2} q^T \left[L^T k' L \right] q = \frac{1}{2} q^T k q$

where $k = \left[L^T k' L \right]$ is the elemental stiffness matrix in global coordinate system.

where L is the transformation matrix

$$L = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \Rightarrow L^T = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \quad \therefore L^T k' = \frac{A_e E_e}{l_e} \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Multiplying the two matrices, $L^T k' = \frac{A_e E_e}{l_e} \begin{bmatrix} l & -l \\ m & -m \\ -l & l \\ -m & m \end{bmatrix}$

$$\therefore k = \left[L^T k' L \right] = \frac{A_e E_e}{l_e} \begin{bmatrix} l & -l \\ m & -m \\ -l & l \\ -m & m \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

Stiffness matrix of truss element $k = \frac{A_e E_e}{l_e} \begin{bmatrix} l^2 & ml & -l^2 & -ml \\ ml & m^2 & -ml & -m^2 \\ -l^2 & -ml & l^2 & ml \\ -ml & -m^2 & ml & m^2 \end{bmatrix}$

Derivation of Element stress matrix of truss element:

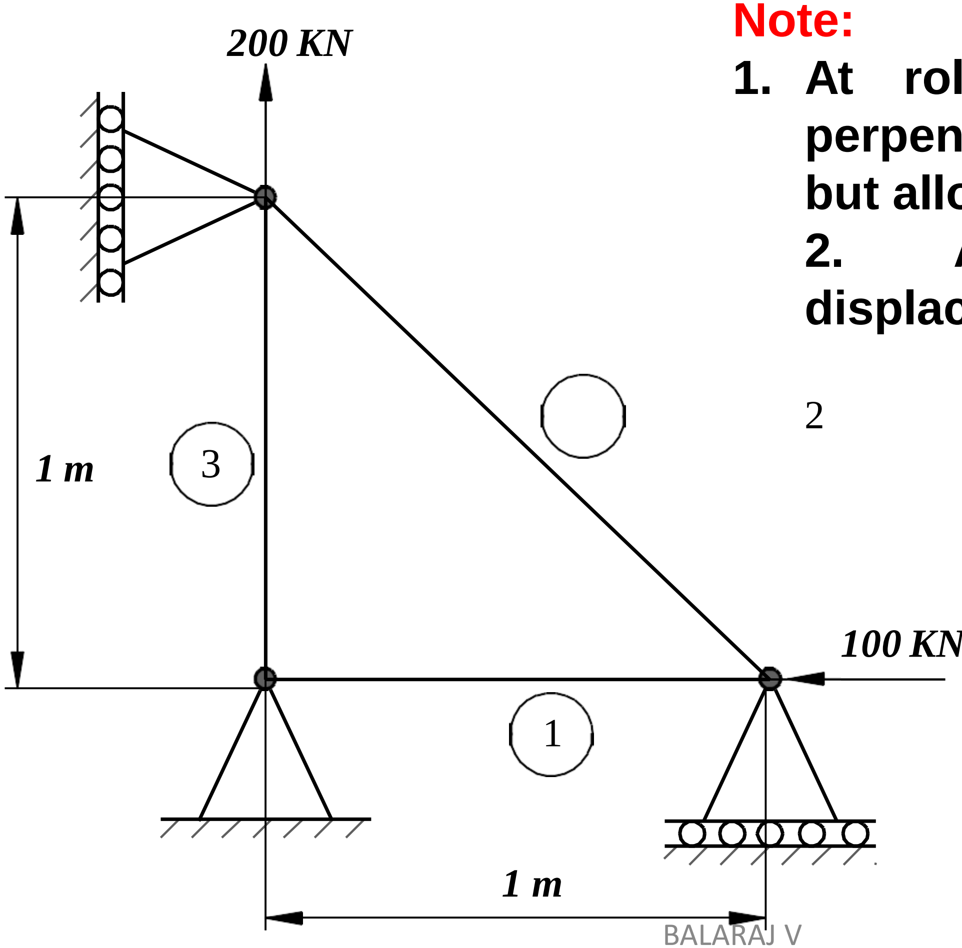
The element stress matrix for a truss element is equivalent to that of 1D bar element. $\sigma = EBq'$ where $B = \frac{1}{l_e}\begin{bmatrix}-1 & 1\end{bmatrix}$

and $q' = \begin{Bmatrix} q_1' \\ q_2' \end{Bmatrix}$ Also $q' = Lq$

$$\therefore \sigma = EBq' = E\,\frac{1}{l_e}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

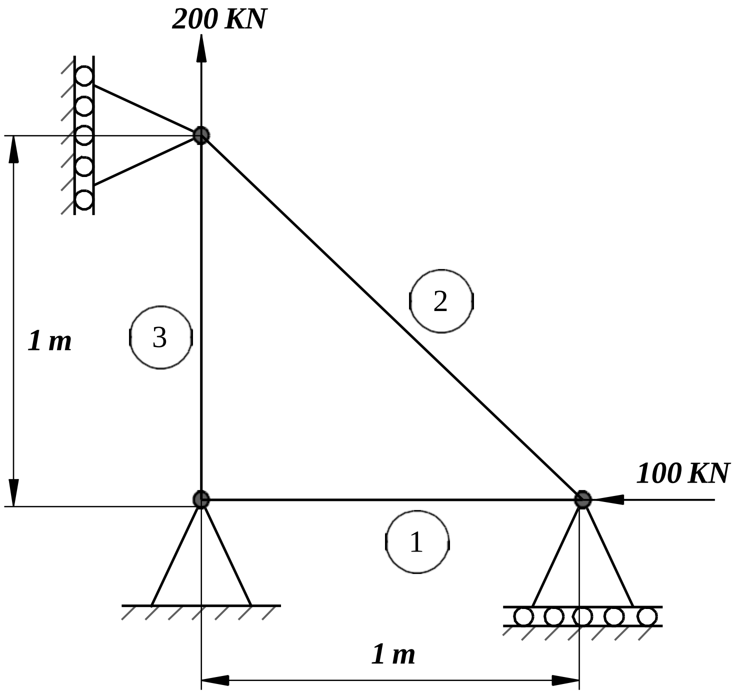
$$\boldsymbol{\sigma = \frac{E}{l_e}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}}$$

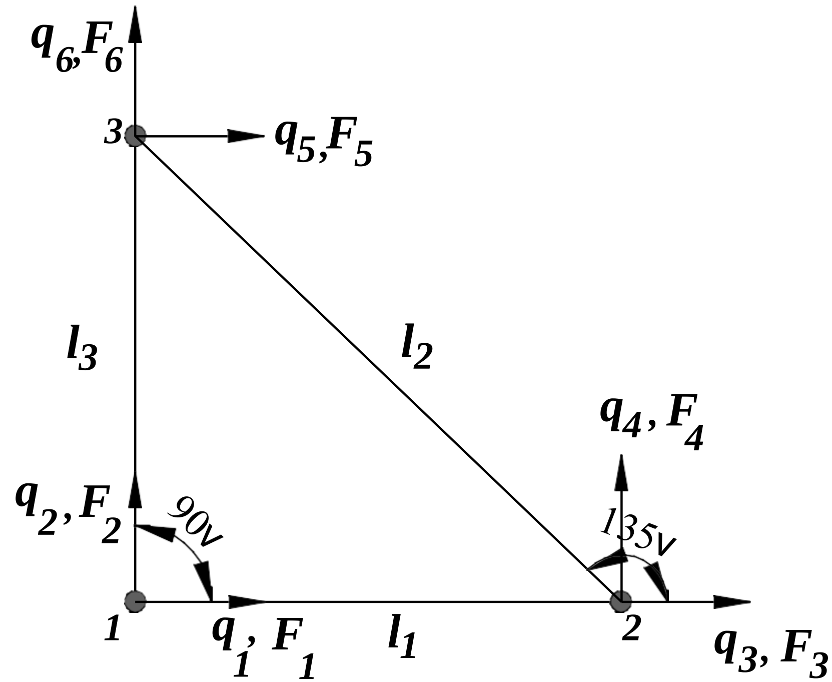
Determine the nodal displacements, stresses & support reactions in the truss segments subjected to point loads as shown in fig. Take E= 70 GPa, A=0.01 m^2.



Note:

1. **At roller supports, the displacement perpendicular to the rollers is constrained but allowed along the rollers.**
2. **At the hinged support, all displacements are constrained.**





Element	θ	l	l^2	m	m^2	lm	Length
1	0^0	1	1	0	0	0	1 m
2	135^0	–0.707	0.5	0.707	0.5	–0.5	1.414 m
3	90^0	0	0	1	1	0	1 m

Element stiffness matrices in global coordinates are given by;

$$\left[k^{(1)}\right] = \frac{AE}{l_1} \begin{bmatrix} l^2 & ml & -l^2 & -ml \\ ml & m^2 & -ml & -m^2 \\ -l^2 & -ml & l^2 & ml \\ -ml & -m^2 & ml & m^2 \end{bmatrix} = \left(\frac{0.01\times 70\times 10^9}{1}\right) \begin{bmatrix} \overset{1}{1} & \overset{2}{0} & \overset{3}{-1} & \overset{4}{0} \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$\left[k^{(1)}\right] = 10^8 \begin{bmatrix} \overset{1}{7} & \overset{2}{0} & \overset{3}{-7} & \overset{4}{0} & \overset{5}{0} & \overset{6}{0} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -7 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

$$
\left[k^{(2)}\right] = \frac{0.01 \times 70 \times 10^{9}}{2\sqrt{2}}
\begin{bmatrix}
 & 3 & 4 & 5 & 6 & \\
 & 1 & -1 & -1 & 1 & 3 \\
 & -1 & 1 & 1 & -1 & 4 \\
 & -1 & 1 & 1 & -1 & 5 \\
 & 1 & -1 & -1 & 1 & 6
\end{bmatrix}
= 2.475 \times 10^{8}
\begin{bmatrix}
 & 3 & 4 & 5 & 6 & \\
 & 1 & -1 & -1 & 1 & 3 \\
 & -1 & 1 & 1 & -1 & 4 \\
 & -1 & 1 & 1 & -1 & 5 \\
 & 1 & -1 & -1 & 1 & 6
\end{bmatrix}
$$

$$
\left[k^{(2)}\right] = 10^{8}
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 2.475 & -2.475 & -2.475 & 2.475 & 3 \\
0 & 0 & -2.475 & 2.475 & 2.475 & -2.475 & 4 \\
0 & 0 & -2.475 & 2.475 & 2.475 & -2.475 & 5 \\
0 & 0 & 2.475 & -2.475 & -2.475 & 2.475 & 6
\end{bmatrix}
$$

$$\left[k^{(3)}\right] = \left(\frac{0.01 \times 70 \times 10^9}{1}\right) \begin{array}{c} \begin{matrix} 1 & 2 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix}$$

$$\left[k^{(3)}\right] = 10^8 \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 & 0 & 7 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

Global stiffness matrix $[K] = k^{(1)} + k^{(2)} + k^{(3)}$

$$
[K] = 10^8
\begin{array}{c}
\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\
\begin{bmatrix}
7 & 0 & -7 & 0 & 0 & 0 \\
0 & 7 & 0 & 0 & 0 & -7 \\
-7 & 0 & 9.475 & -2.475 & -2.475 & 2.475 \\
0 & 0 & -2.475 & 2.475 & 2.475 & -2.475 \\
0 & 0 & 2.475 & 2.475 & 2.475 & -2.475 \\
0 & -7 & 2.475 & -2.475 & -2.475 & 9.475
\end{bmatrix}
\end{array}
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}
$$

Global load vector is $\{F\} = 10^3 \begin{Bmatrix} 0 \\ 0 \\ -100 \\ 0 \\ 0 \\ 200 \end{Bmatrix}$

The equation of equilibrium is KQ = F

$$10^8\begin{bmatrix} 7 & 0 & -7 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & -7 \\ -7 & 0 & 9.475 & -2.475 & -2.475 & 2.475 \\ 0 & 0 & -2.475 & 2.475 & 2.475 & -2.475 \\ 0 & 0 & -2.475 & 2.475 & 2.475 & -2.475 \\ 0 & -7 & 2.475 & -2.475 & -2.475 & 9.475 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = 10^3\begin{Bmatrix} 0 \\ 0 \\ -100 \\ 0 \\ 0 \\ 200 \end{Bmatrix}$$

(Rows and columns numbered 1–6.)

Imposing the boundary conditions $q_1 = q_2 = q_4 = q_5 = 0$

(@ roller supports, normal displacements are constrained &
@ hinged supports, all displacements are constrained)

& using elimination approach, $10^8\begin{bmatrix} 9.475 & 2.475 \\ 2.475 & 9.475 \end{bmatrix}\begin{Bmatrix} q_3 \\ q_6 \end{Bmatrix} = 10^3\begin{Bmatrix} -100 \\ 200 \end{Bmatrix}$

Solving, $\mathbf{q_3 = -0.17 \times 10^{-5}\ m,\ q_6 = 0.25 \times 10^{-5}\ m}$

Stresses in elemnts : In element 1.

$$\sigma^{(1)} = \frac{E}{l_e}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\begin{Bmatrix}q_1\\q_2\\q_3\\q_4\end{Bmatrix} = \frac{70\times10^9}{1}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}0\\0\\q_3\\0\end{Bmatrix}$$

Solving, $\sigma^{(1)} = 70\times10^9\times(-0.17\times10^{-5})$ = $\mathbf{0.119\times10^6\ N/m^2}$

In element 2, $\sigma^{(2)} = \frac{E}{l_e}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\begin{Bmatrix}q_3\\q_4\\q_5\\q_6\end{Bmatrix}$

$$= \frac{70\times10^9}{1.414}\begin{bmatrix}0.707 & -0.707 & -0.707 & 0.707\end{bmatrix}\begin{Bmatrix}q_3\\0\\0\\q_6\end{Bmatrix}$$

Solving, $\sigma^{(2)} = \frac{70\times10^9}{1.414}\times0.707\times10^{-5}(-0.17+0.25)$ = $\mathbf{0.028\times10^6\ N/m^2}$

Stresses in elemnts : In element 3.

$$\sigma^{(3)} = \frac{E}{l_e}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_5 \\ q_6 \end{Bmatrix} = \frac{70\times10^9}{1}\begin{bmatrix}0 & -1 & 0 & 1\end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ q_6 \end{Bmatrix}$$

Solving, $\sigma^{(3)} = 70\times10^9\times(0.25\times10^{-5})$ = $\mathbf{0.175\times10^6\ N/m^2}$

Reactions at supports : R = KQ - F

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{Bmatrix} = 10^8 \begin{bmatrix} 7 & 0 & -7 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & -7 \\ -7\ 0 & 9.475 & -2.475 & & -2.475\ 2.475 & \\ 0\ 0 & -2.475 & 2.475 & 2.475 & -2.475 & \\ 0\ 0 & 2.475 & 2.475 & 2.475 & -2.475 & \\ 0 & -7 & 2.475 & -2.475 & 9.475 & \\ & -2.475 & & & & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ 0 \\ 0 \\ q_6 \end{Bmatrix} - 10^3 \begin{Bmatrix} 0 \\ 0 \\ -100 \\ 0 \\ 0 \\ 200 \end{Bmatrix}$$

(Column labels 1 2 3 4 5 6; row labels 1 2 3 4 5 6)

$$R_1 = -7\times10^8(-0.17\times10^{-5}) - 0 = \boldsymbol{1190\ N}$$

$$R_2 = -7\times10^8(0.25\times10^{-5}) - 0 = \boldsymbol{-1750\ N}$$

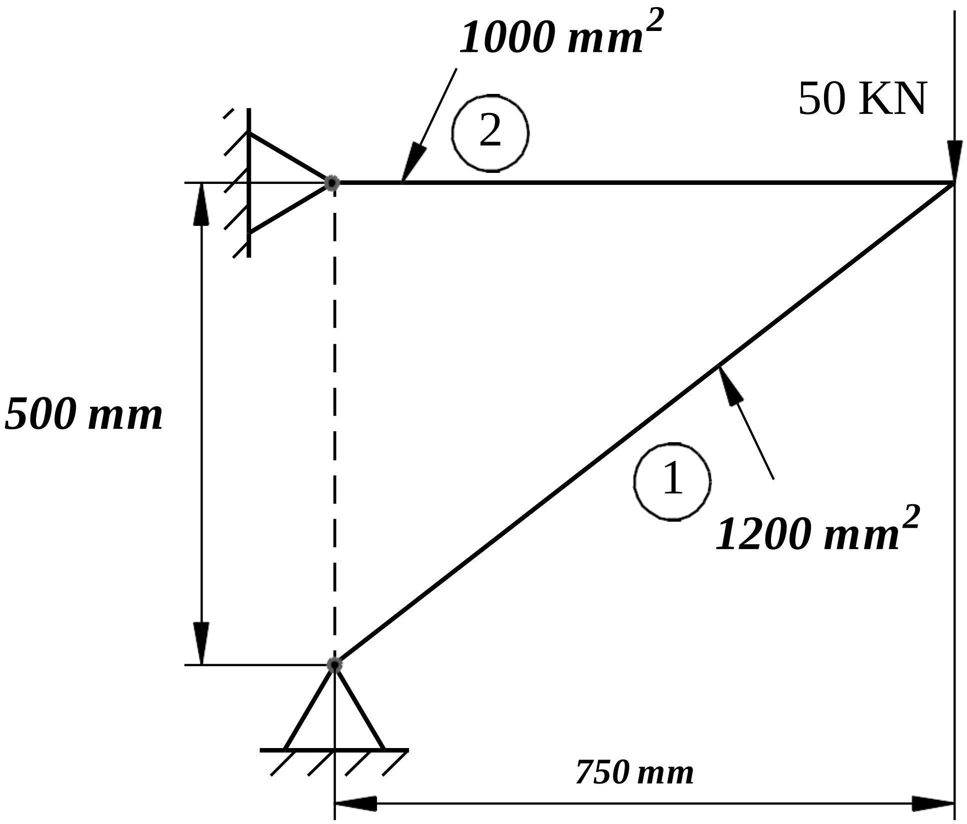
$$R_3 = 10^8\left[9.475\times(-0.17\times10^{-5}) + 2.475(0.25\times10^{-5})\right] - (-100)10^3 = \boldsymbol{99008\ N}$$

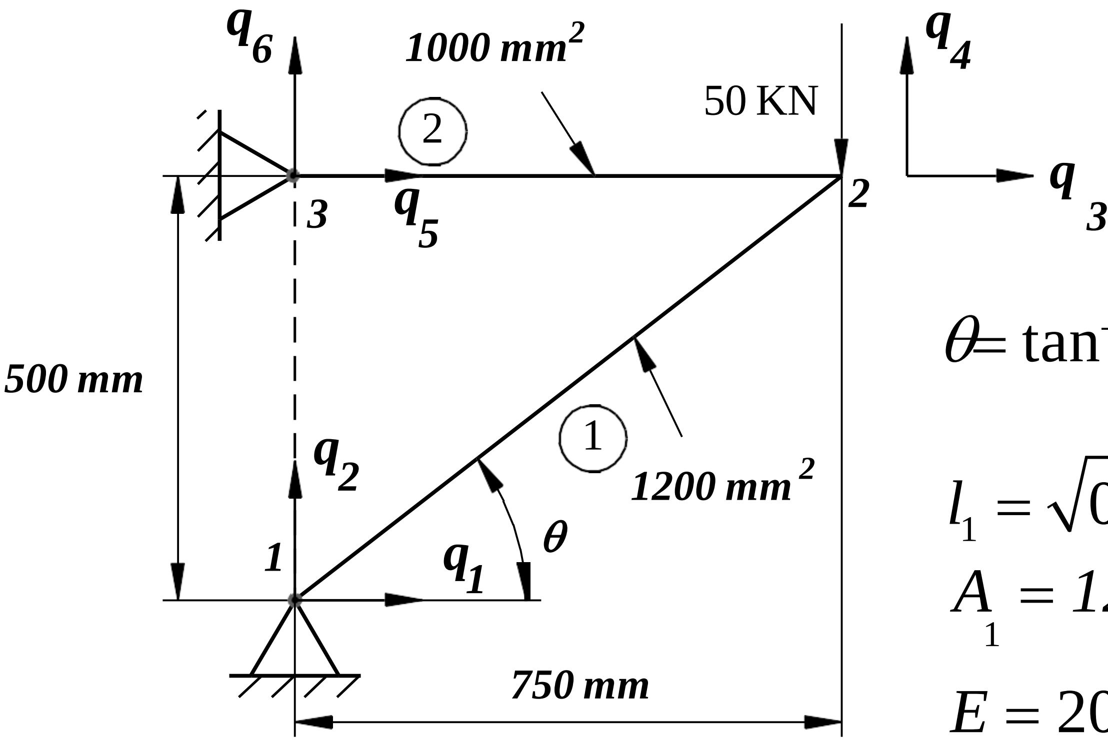
$$R_4 = 10^8\left[-2.475\times(-0.17\times10^{-5}) - 2.475\times(0.25\times10^{-5})\right] - 0 = \boldsymbol{-198\ N}$$

$$R_5 = 10^8\left[2.475\times(-0.17\times10^{-5}) - 2.475(0.25\times10^{-5})\right] - 0 = \boldsymbol{1039.5\ N}$$

$$R_6 = 10^8\left[2.475\times(-0.17\times10^{-5}) + 9.475\times(0.25\times10^{-5})\right] - 200\times10^3 = \boldsymbol{-198052\ N}$$

For a two element truss member shown in fig, determine the nodal displacements and stress in each member. Take E=200 Gpa.





$$\theta = \tan^{-1}\left(\frac{500}{750}\right) = 33.7^0$$

$$l_1 = \sqrt{0.75^2 + 0.5^2} = 901.4\ mm$$

$$A_1 = 1200\ mm^2, A_2 = 1000\ mm^2$$

$$E = 200 \times 10^3\ N/mm^2$$

Element	θ	*l*	l^2	*m*	m^2	*lm*	*Length*
1	33.7^0	0.832	0.692	0.555	0.308	0.462	901.4 *mm*
2	180^0	-1	1	0	0	0	750 *mm*

Element stiffness matrices in global coordinates are given by;

$$\left[k^{(1)}\right] = \frac{A_1 E}{l_1}\begin{bmatrix} l^2 & ml & -l^2 & -ml \\ ml & m^2 & -ml & -m^2 \\ -l^2 & -ml & l^2 & ml \\ -ml & -m^2 & ml & m^2 \end{bmatrix} = \left(\frac{1200\times 200\times 10^3}{901.4}\right)\begin{bmatrix} 0.692 & 0.462 & -0.692 & -0.462 \\ 0.462 & 0.308 & -0.462 & -0.308 \\ -0.692 & -0.462 & 0.692 & 0.462 \\ -0.462 & -0.308 & 0.462 & 0.308 \end{bmatrix}$$

(columns labelled 1, 2, 3, 4; rows labelled 1, 2, 3, 4)

$$\left[k^{(1)}\right] = 10^5\begin{bmatrix} 1.84 & 1.23 & -1.84 & -1.23 & 0 & 0 \\ 1.23 & 0.82 & -1.23 & -0.82 & 0 & 0 \\ -1.84 & -1.23 & 1.84 & 1.23 & 0 & 0 \\ -1.23 & -0.82 & 1.23 & 0.82 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(columns labelled 1, 2, 3, 4, 5, 6; rows labelled 1, 2, 3, 4, 5, 6)

$$\left[k^{(2)}\right] = \frac{A_2E}{l_2}\begin{bmatrix} l^2 & ml & -l^2 & -ml \\ ml & m^2 & -ml & -m^2 \\ -l^2 & -ml & l^2 & ml \\ -ml & -m^2 & ml & m^2 \end{bmatrix} = \left(\frac{1000\times 200\times 10^3}{750}\right)\begin{bmatrix} \overset{3}{1} & \overset{4}{0} & \overset{5}{-1} & \overset{6}{0} \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

$$\left[k^{(2)}\right] = 10^5\begin{bmatrix} \overset{1}{0} & \overset{2}{0} & \overset{3}{0} & \overset{4}{0} & \overset{5}{0} & \overset{6}{0} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.67 & 0 & -2.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2.67 & 0 & 2.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

Global stiffness matrix $[K] = k^{(1)} + k^{(2)}$

$$[K] = 10^5 \begin{bmatrix} 1.84 & 1.23 & -1.84 & -1.23 & 0 & 0 \\ 1.23 & 0.82 & -1.23 & -0.82 & 0 & 0 \\ -1.84 & -1.23 & 4.5 & 1.23 & -2.67 & 0 \\ -1.23 & -0.82 & 1.23 & 0.82 & 0 & 0 \\ 0 & 0 & -2.67 & 0 & 2.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(Columns numbered 1–6 above; rows numbered 1–6 at right.)

Global load vector is $\{F\} = 10^3 \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -50 \\ 0 \\ 0 \end{Bmatrix}$

The equation of equilibrium is KQ=F

$$10^5\begin{bmatrix} 1.84 & 1.23 & -1.84 & -1.23 & 0 & 0 \\ 1.23 & 0.82 & -1.23 & -0.82 & 0 & 0 \\ -1.84 & -1.23 & 4.5 & 1.23 & -2.67 & 0 \\ -1.23 & -0.82 & 1.23 & 0.82 & 0 & 0 \\ 0 & 0 & -2.67 & 0 & 2.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = 10^3 \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -50 \\ 0 \\ 0 \end{Bmatrix}$$

(Rows and columns labelled 1–6.)

Imposing the boundary conditions $q_1 = q_2 = q_5 = q_6 = 0$

(At pin joints,(hinged supports) all displacements are constrained)

& using elimination approach, $10^5\begin{bmatrix} 4.5 & 1.23 \\ 1.23 & 0.82 \end{bmatrix}\begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = 10^3\begin{Bmatrix} 0 \\ -50 \end{Bmatrix}$

Solving, **$q_3 = 0.2825$ mm, $q_4 = -1.033$ mm**

Stresses in elemnts : In element 1. $\sigma^{(1)} = \frac{E}{l_1}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$

$$\sigma^{(1)} = \frac{200\times10^3}{901.4}\begin{bmatrix} -0.832 & -0.555 & 0.832 & 0.555 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.2825 \\ -1.033 \end{Bmatrix}$$

Solving, $\sigma^{(1)} = 221.88\times\left[(0.832 \times 0.2825 + 0.555(-1.033))\right]$ = ***-75.06 N / mm²***

In element 2, $\sigma^{(2)} = \frac{E}{l_2}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\begin{Bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \frac{200\times10^3}{750}\begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}\begin{Bmatrix} 0.2825 \\ -1.033 \\ 0 \\ 0 \end{Bmatrix}$

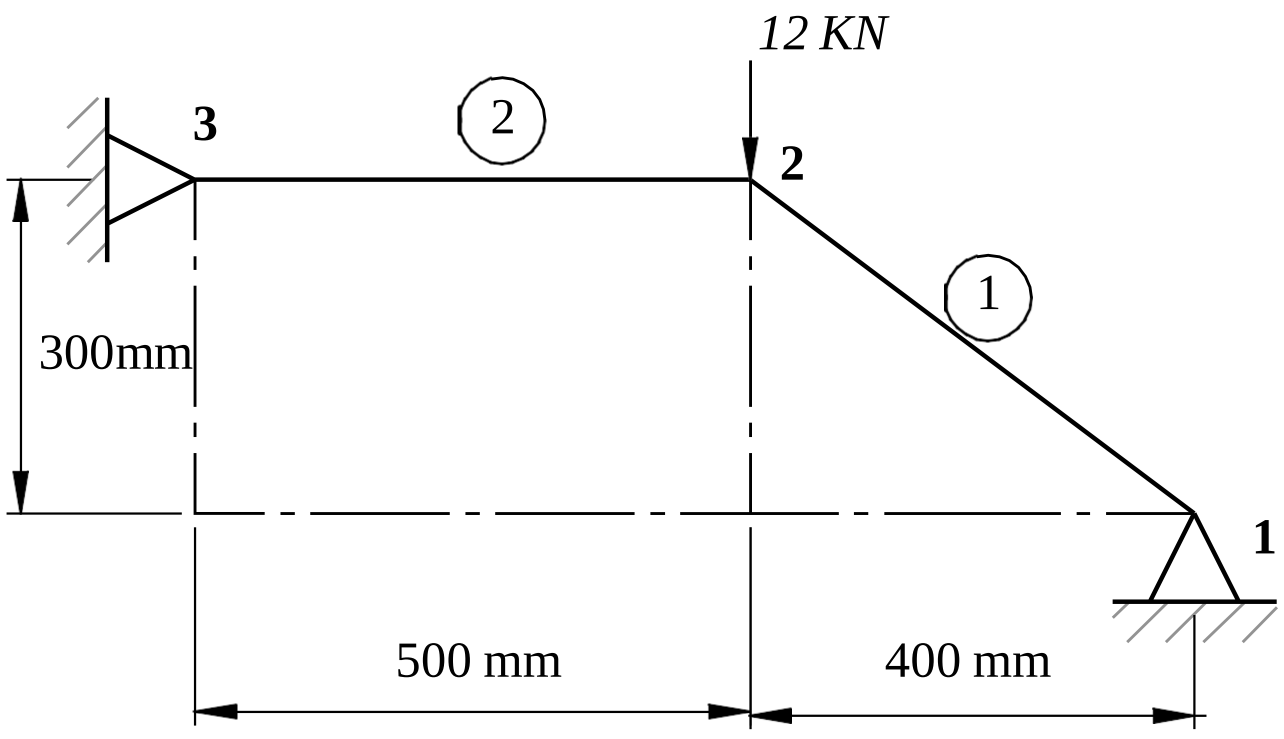
Solving, $\sigma^{(2)} = 266.67\times\left[(1\times0.2825+0\right]$ = ***75.33 N / mm²***

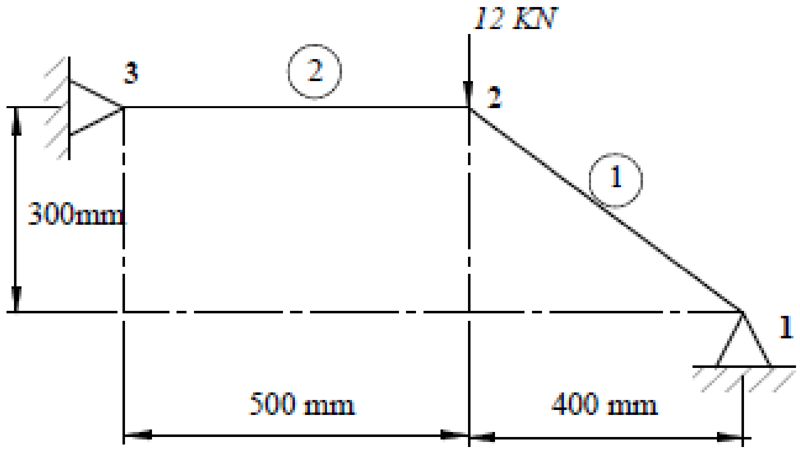
Reactions at supports : R = KQ - F

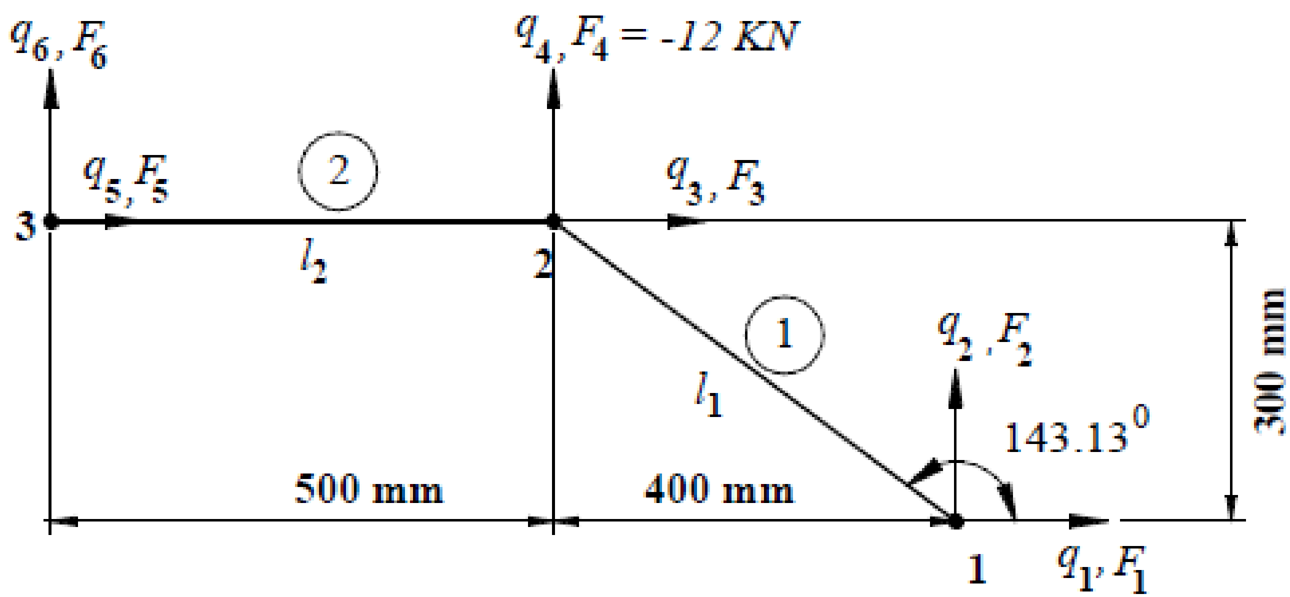
$$\Rightarrow \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{Bmatrix} = 10^5 \begin{bmatrix} \overset{1}{1.84} & \overset{2}{1.23} & \overset{3}{-1.84} & \overset{4}{-1.23} & \overset{5}{0} & \overset{6}{0} \\ 1.23 & 0.82 & -1.23 & -0.82 & 0 & 0 \\ -1.84 & -1.23 & 4.5 & 1.23 & -2.67 & 0 \\ -1.23 & -0.82 & 1.23 & 0.82 & 0 & 0 \\ 0 & 0 & -2.67 & 0 & 2.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.2825 \\ -1.033 \\ 0 \\ 0 \end{Bmatrix} - 10^3 \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -50 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{Bmatrix} = \begin{Bmatrix} 75039 \\ 49959 \\ 66 \\ 41.5 \\ 75428 \\ 0 \end{Bmatrix}$$

Obtain the nodal displacements and reactions at supports in the truss shown in fig. Take E=200 Gpa, A=200 mm².




12 KN
3
2
2
1
300mm
1
500 mm
400 mm


q6, F6
q4, F4 = -12 KN
q5, F5
2
q3, F3
3
l2
2
1
q2, F2
l1
143.13°
500 mm
400 mm
300 mm
1
q1, F1

$$l_1 = 500 \ \ mm, \ \ l_2 = \sqrt{300^2 + 400^2} = 500 \ mm$$

Direction Cosines:

Element	θ	l	l^2	m	m^2	lm	*Length*
1	143.13^0	-0.8	0.64	0.6	0.36	0.48	500
2	180^0	-1	1	0	0	0	500

Element stiffness matrices in global coordinates are given by;

$$\left[k^{(1)}\right] = \frac{A_1 E}{l_1}\begin{bmatrix} l^2 & ml & -l^2 & -ml \\ ml & m^2 & -ml & -m^2 \\ -l^2 & -ml & l^2 & ml \\ -ml & -m^2 & ml & m^2 \end{bmatrix} = \frac{200 \times 200 \times 10^3}{500}\begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix}$$

$$\left[k^{(1)}\right]=10^4\begin{bmatrix} 5.12 & -3.84 & -5.12 & -0.48 \\ -3.84 & 2.88 & 3.84 & -2.88 \\ -5.12 & 3.84 & 5.12 & -3.84 \\ 3.84 & -2.88 & -3.84 & 2.88 \end{bmatrix}\begin{matrix}1\\2\\3\\4\end{matrix}$$

(columns: 1, 2, 3, 4)

$$\text{Similarly } \left[k^{(2)}\right]=10^4\begin{bmatrix} 8 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix}3\\4\\5\\6\end{matrix}$$

(columns: 3, 4, 5, 6)

$$\text{Global stiffness matrix } [K] = 10^4 \begin{bmatrix} \overset{1}{5.12} & \overset{2}{-3.84} & \overset{3}{-5.12} & \overset{4}{3.84} & \overset{5}{0} & \overset{6}{0} \\ -3.84 & 2.88 & 3.84 & -2.88 & 0 & 0 \\ -5.12 & 3.84 & 13.12 & -3.84 & -8 & 0 \\ 3.84 & -2.88 & -3.84 & 2.88 & 0 & 0 \\ 0 & 0 & -8 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

Global load vector $[F] = 10^3 \{0 \quad 0 \quad -12 \quad 0 \quad 0 \quad 0\}^T$

$\therefore$ Equilibrium equation is $[K]\{q\} = F$ where $\{q\} = \{q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5 \quad q_6\}^T$

Applying bc's $q_1 = q_2 = q_5 = q_6 = 0$, the equilibrium equation reduces to;

$$10^4 \begin{bmatrix} 13.12 & -3.84 \\ -3.84 & 2.88 \end{bmatrix} \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = 0 \Rightarrow \boldsymbol{q_3 = -0.2\ mm, q_4 = -0.683\ mm}$$

Also the reactions are $R = [K]\{q\} - F \Rightarrow \boldsymbol{R_1 = 15987\ N, R_2 = 11990\ N, R_5 = 16000\ N}$

Element Stresses

$$\sigma_1 = \frac{E}{l_1}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\begin{Bmatrix}q_1\\q_2\\q_3\\q_4\end{Bmatrix} = \frac{2\times10^5}{500}\begin{bmatrix}0.8 & -0.6 & -0.8 & 0.6\end{bmatrix}\begin{Bmatrix}0\\0\\-0.2\\-0.683\end{Bmatrix} = \textbf{-99.992 MPa}$$

$$\sigma_2 = \frac{E}{l_2}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\begin{Bmatrix}q_3\\q_4\\q_5\\q_6\end{Bmatrix} = \frac{2\times10^5}{500}\begin{bmatrix}0.8 & -0.6 & -0.8 & 0.6\end{bmatrix}\begin{Bmatrix}-0.2\\-0.983\\0\\0\end{Bmatrix} = \textbf{-80 MPa}$$