FINITE ELEMENT METHODS

Subject code:18ME61

Credits : 4
Theory : 3 hours ,Tutorial : 2 hours

CIE Marks: 40 , SEE Marks:60, Total Mark:100

April 2021
One Solution for All Engineering Problems

Course Learning Objectives

1. The concept of finite element methods in engineering design
2. Learning selection of elements and boundary conditions for analysis
3. Discuss 1D and 2D solutions for simple components such as bars, trusses and beams
4. Apply finite element solutions to structural, thermal, dynamic problem to develop the knowledge and skills needed to effectively evaluate finite element analyses.

Course Outcome

1. Define the fundamentals of finite element methods
2. Develop the knowledge to analyse structures in static and dynamic conditions
3. Assess numerical techniques for solving engineering problems
4. Formulate finite element model to implement industrial projects

Reference Books

1. Hutton, 'Fundamentals of FEM', Tata McGraw Hill Education Pvt. Ltd. 2005, ISBN:0070601224
2. Daryl L Logon, 'First Course in Finite Element Methods', 5th Edn, Thomson Brooks, 2011, ISBN-10: 0495668257.
3. George R Buchanan, 'Finite Element Analysis', Tata Mcgraw Hill, 2004, ISBN: 0070087148.
4. **T.R.Chandrapatla, A D Belegundu, 'Introduction to FE in Engineering', 3rd Edn., Prentice hall, 2004.**

Sl No	Title of the Book	Name of the Author/s	Name of the Publisher	Edition and Year
Textbook/s				
1	A first course in the Finite Element Method	Logan, D. L	Cengage Learning	6th Edition2016
2	Finite Element Method in Engineering	Rao, S. S	Pergaman Int. Library of Science	5th Edition2010
3	Finite Elements in Engineering	Chandrupatla T. R	PHI	2nd Edition2013
Reference Books				
1	Finite Element Method	J.N.Reddy	McGraw -Hill International Edition	
2	Finite Elements Procedures	Bathe K. J	PHI	
3	Concepts and Application of Finite Elements Analysis	Cook R. D., et al	Wiley & Sons	4th Edition2003

Module-1:Introduction to FEM

Introduction to Finite Element Method:

- General description of FEM,
- Steps involved in FEM,
- Engineering applications of FEM, Advantages of FEM,

Boundary conditions:

- Homogeneous and non-homogeneous for structural
- Heat transfer and fluid flow problems.
- Potential energy method,
- Rayleigh Ritz Method,
- Galerkin's Method,
- Displacement method of finite element formulation.
- Convergence criteria, Discretisation process.

Module-1:Introduction to FEM

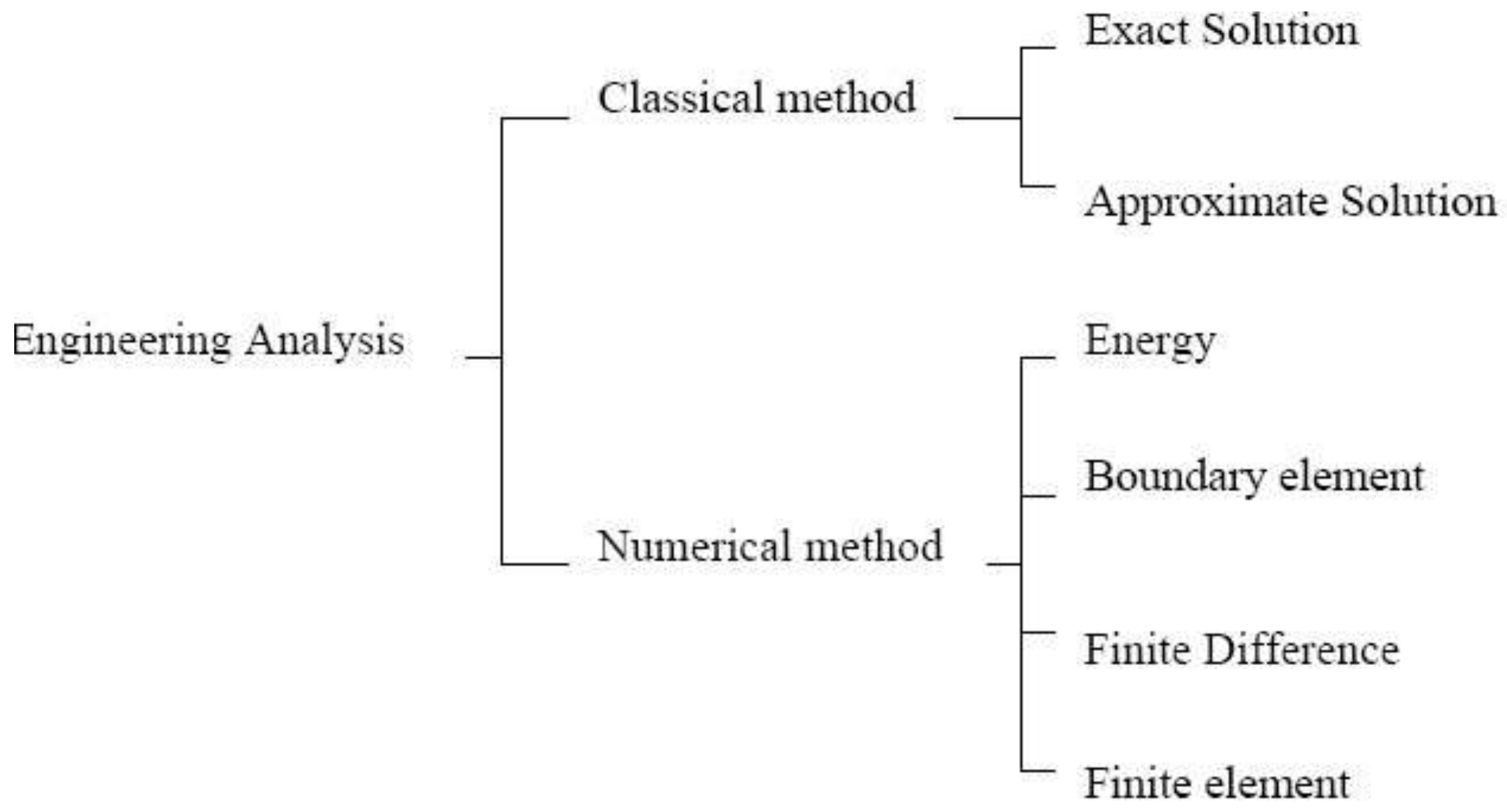
- **Types of elements:** 1D, 2D and 3D, Node numbering, Location of nodes.
- Strain- displacement relations, Stress-strain relations, Plain stress and Plain strain conditions, temperature effects.

Interpolation models:

- Simplex, complex and multiplex elements
- Linear interpolation polynomials in terms of global coordinates 1D, 2D, 3D Simplex Elements.

1.1 General Description of FEM

Methods of Solution



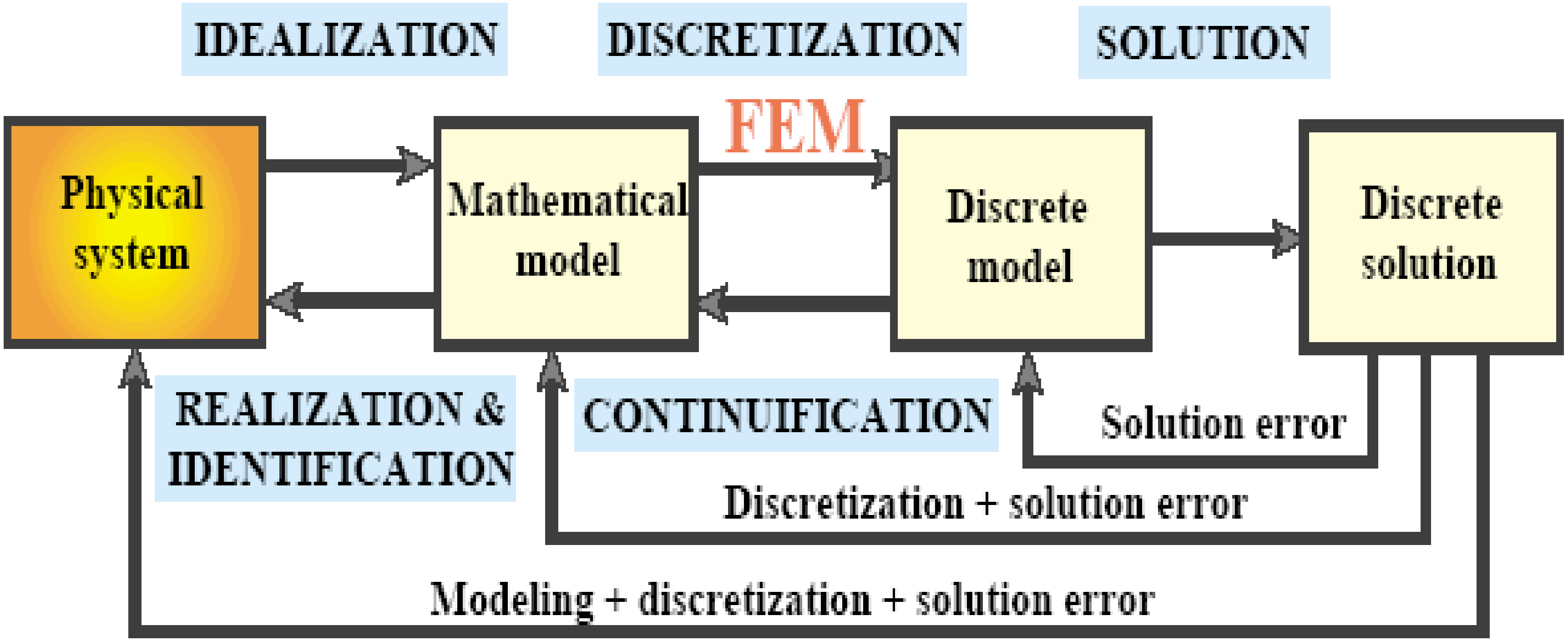
1.1 General Description of FEM

- **Classical Method:** They offer a high degree of insight, but the problems are difficult or impossible to solve for anything but simple geometries and loadings.
- **Numerical methods**
- Energy**:** Minimize an expression for the potential energy of the structure over the whole domain (Rayleigh Ritz Method)
- Boundary element: Approximates functions satisfying the governing differential equations not the boundary conditions (Galerkin's Method)
- Finite difference: Replaces governing differential equations and boundary conditions with algebraic finite difference equations (Newton Raphson method)
- Finite element: Approximates the behavior of an irregular, continuous structure under general loadings and constraints with an assembly of discrete elements.

1.1 General Description of FEM

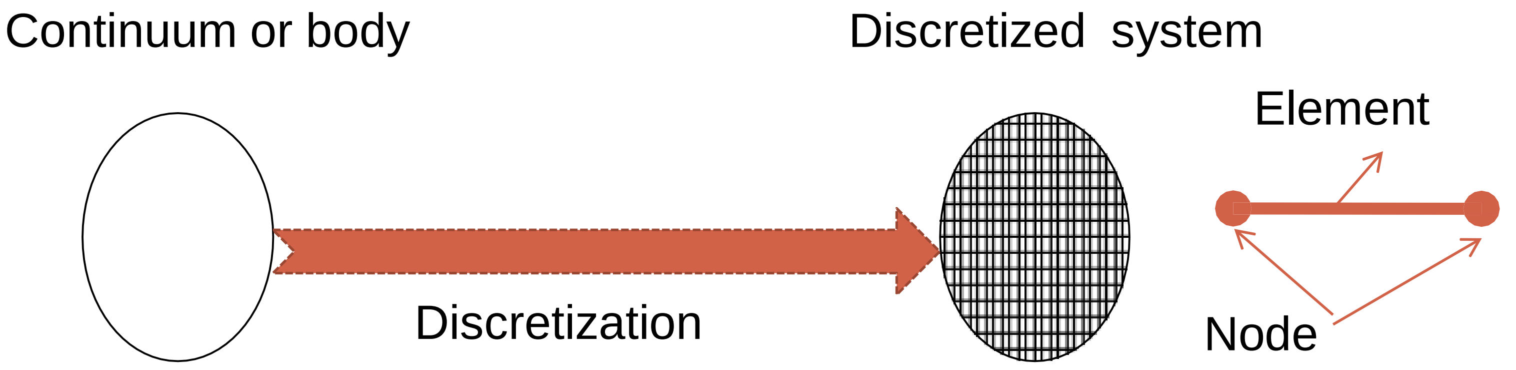
- **Mathematical Model:** A model is a symbolic device built to simulate and predict aspects of behavior of a system or Abstraction of physical reality (eg $x^2 + y^2 = a^2$ for Circle)
- **Finite element method (FEM):** It is a numerical technique for finding approximate solutions to boundary value problems for partial differential equations.
- FEM sub divides a large problem into smaller, simpler, parts, called finite elements.
- The simple equations that model these finite elements are then assembled into a larger system of equations that models the entire problem.
- **Implicit Modelling:** It consists of using existent pieces of abstraction and fitting them into the particular situation.
- **Explicit Modeling:** It consists of building the model from scratch

1.1 General Description of FEM



1.1 General Description of FEM

- Discretization: Modeling a body by dividing it into an equivalent system of finite elements interconnected at a finite number of points on each element called nodes.



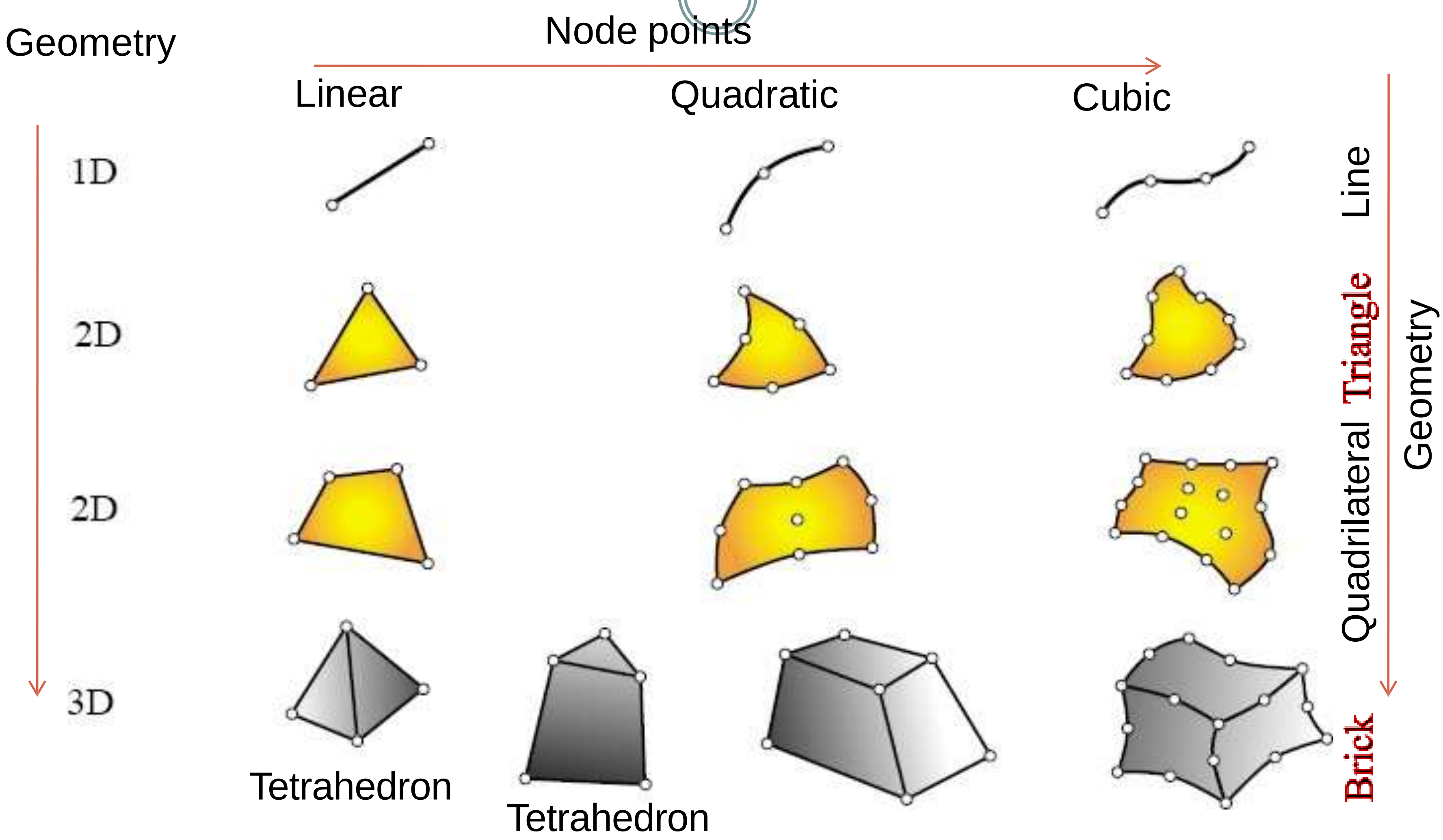
- Node: interconnected at points common to two or more elements and / or boundary lines and / or surfaces.
- Element body by dividing it into an equivalent system of many smaller bodies or units

1.1 General Description of FEM

Elements are defined by the following properties

1. **Dimensionality: 1D, 2D and 3D**
2. **Nodal Points : Linear (a+bx), Quadratic (a+bx+cx^2) and so on**
3. **Geometry: Line, Triangle, Quadrilateral, Brick, tetrahedron, etc.**
4. **Degrees of Freedom: Linear (1D, 2D &3D), rotation(M_x M_y & M_z)**
5. **Nodal Forces: Linear (1D, 2D &3D), rotation(M_x M_y & M_z)**

1.1 General Description of FEM



1.1 General Description of FEM

Available Commercial FEM Software Packages

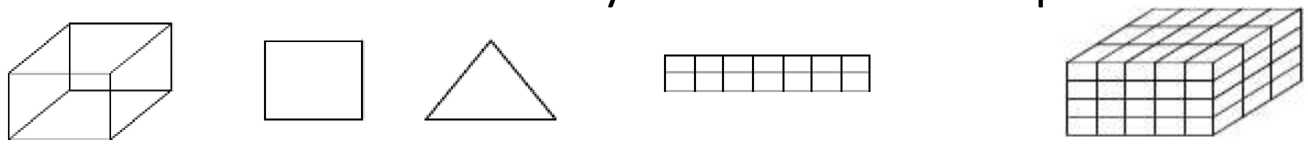
- *ANSYS* (General purpose, PC and workstations)
- *SDRC/I-DEAS* (Complete CAD/CAM/CAE package)
- *NASTRAN* (General purpose FEA on mainframes)
- *ABAQUS* (Nonlinear and dynamic analyses)
- *COSMOS* (General purpose FEA)
- *ALGOR* (PC and workstations)
- *PATRAN* (Pre/Post Processor)
- *HyperMesh* (Pre/Post Processor)
- *Dyna-3D* (Crash/impact analysis)
- ...

1.2 Steps involved in FEM

STEP I: DISCRETIZATION OF THE STRUCTURE

- The continuum is separated by imaginary lines of surfaces into a number of finite elements.

- The number, type, size and the arrangements of the elements have to be decided based on the accuracy of the solution required.



`Discretize and select the element types

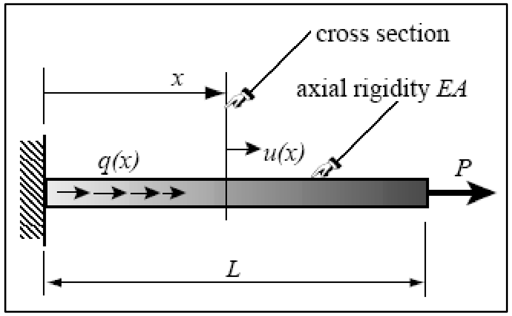
(a)element type 1D line element 2D element

3D brick element

total number of element (mesh) 1D, 2D,3D.

1.2 Steps involved in FEM

Step 2: Selection of a proper interpolation function or Displacement model and check for boundary condition



Since the displacement solution of a complex structure under any specified load conditions cannot be predicted exactly, we assume some suitable solution within element to approximate the unknown solution.

$$1D: \varepsilon x = \frac{du}{dx} \qquad \sigma = E\varepsilon$$

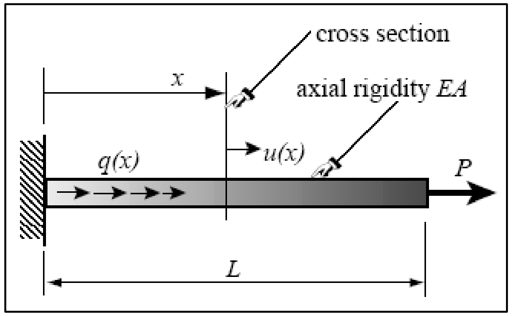
The assumed solution must be simple from computational point of view.

1D line element: u=ax+b $\quad [K]^e\{d\}^e = \{F\}^e$

BC-1 at x = 0 then u = 0 then b = 0 Then u = ax,

1.2 Steps involved in FEM

Step 2: Selection of a proper interpolation function or Displacement model and check for boundary condition



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Steps involved in FEM

SSTEP 3: Derivation of element stiffness matrices and load vectors

➢From the assumed displacement model, the stiffness matrix [k_e] and the load vector [p_e], of element ‘e’ are to be derived by using either equilibrium conditions or a suitable variational principle.

Form the element stiffness matrix and equations

(a) Direct equilibrium method

(b) Work or energy method

(c) Method of weight Residuals

1.2 Steps involved in FEM

Step 5: Form the system equation

Assemble the element equations to obtain global system equation and introduce boundary conditions

$$[K]\{d\} = \{F\}$$

[k] = assembled stiffness matrix ,

{q} = Vector of nodal displacements ,

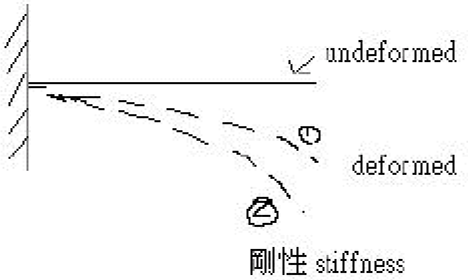
{p} = Vector of nodal forces for the complete structure.

Step 6: Solve the system equations (solve constant)

a. Elimination method : Gauss's method

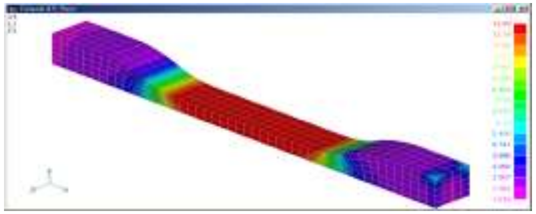
b. Penalty approach method

Step 7: Interpret the results (Find deformation, Stress, strain, etc.)



a. deformation plot

b. stress contour



K u = F			
	Property [K]	Behavior [u]	Action [F]
Elastic	Stiffness	Displacement	Force (load)
Thermal	Conductivity	Temperature	Heat source
Fluid	Viscosity	Velocity	Body Force
Electrostatic	Dielectric Permeability	Electric Potential	Charge

1.3 Engineering Applications of FEM

Structural Problem	Non-structural Problem
➢ Stress Analysis ✓ Truss & Frame analysis ✓ Stress concentrated problem ➢ Buckling problem ➢ Vibration Analysis ➢ Impact Problem ➢ Static / Dynamic ➢ Linear / Nonlinear	➢ Heat Transfer ➢ Fluid Mechanics ➢ Electric or Magnetic Potential ➢ Soil Mechanics ➢ Acoustics ➢ Biomechanics
Mechanical, Aerospace, Civil, Automotive, Electronics, etc	

1.3 Engineering Applications of FEM

Displacement Models

In FEA the first step is to discritise the given continuum into smaller nuimber of parts called elements. The displacement variation for theses elements are unknown hence a trail function is assumed for the displacement of an element. They are two functions

Polynomial: polynomial functions is best suited for the displacement model since mathematical calculation are simpler. A polynomial function or a displacement models of nth order will give an exact solution for all practical purposes the function is truncated to a finite order so as to simplify the calculations involved. Hence the solution obtained is an approximate one.

$$w(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^4 + \ldots + a_n x^n$$

Trigonometric displacement :

$$= c_1 Sin\left(\frac{\pi x}{L}\right) + c_1 Sin\left(\frac{3\pi x}{L}\right)$$

1.4 Advantages of FEM

FEM can handle

- Irregular Boundaries (straight, curve, etc)
- General Loads (static, dynamic, impact, ...)
- Different Materials (metal, alloys, polymers, composites, ...)
- Boundary Conditions (simple supported, fixed, partially fixed)
- Variable Element Size (nano to Km)
- Easy Modification (changing is very easy)
- Nonlinear Problems (Geometric or Material)

1.4 Advantage and dis of FEM

Advantage	Disadvantage
Can readily handle very complex geometry	A general closed-form solution, which would permit one to examine system response to changes in various parameters, is not produced.
Can handle a wide variety of engineering problems (Solid mechanics, Dynamics, Heat problems, Fluids, Electrostatic problems…)	The FEM obtains only "approximate“ solutions.
Can handle complex restraints (Indeterminate structures can be solved)	The FEM has "inherent" errors.
Can handle complex loading ➢ Nodal load -point loads ➢ Element load (pressure, thermal, inertial forces) ➢ Time or frequency dependent loading	Mistakes by users can be fatal

2.1 Potential Energy

Work Potential: A body is subjected to three varieties of forces a) Body force "f" (b) Traction "T" and (c) point load

a) Work potential due to Body force = $$W = \int_0^L u^T dv\, f$$

b) Work potential due to Traction = $$W = \int_0^L u^T dv T$$

c) Work potential due to point load = $$W = \sum u_i p_i$$

Strain energy is the energy stored by a system undergoing deformation. When the load is removed, strain energy as:

$$U = \int_0^L \frac{1}{2} \sigma^T \varepsilon \, dv$$

2.1 Potential Energy

Potential Energy: is the sum of the elastic strain energy, stored in the deformed body and the potential energy associated to the applied forces

= SE – WP (due to body force) – WP(due to traction) – WP due to point load)

$$\Pi = \int_0^L \frac{1}{2}\,\sigma^T \varepsilon\, dv - \int_0^L u^T dv\, f - \int_0^L u^T dv T - \sum u_i p_i$$

Minimum potential Energy

For conservative structural systems, of all the kinematically admissible deformations, those corresponding to the equilibrium state extremize (i.e., minimize or maximize) the total potential energy. If the extremum is a minimum, the equilibrium state is stable. $\prod = 0$

Kinematically Admissible: these are any reasonable displacement that you can think of that satisfy the displacement boundary conditions of the original problem

Problems will be solved In class room

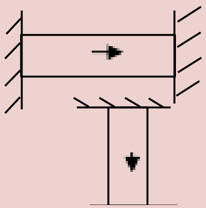
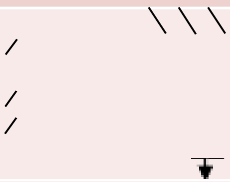
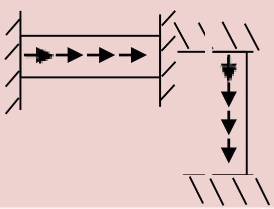
1.5 Rayleigh Ritz Method

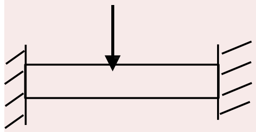
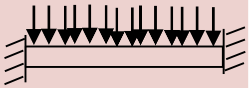
This is an analytical method of determining approximate solution for a given problem, it is a type of continuum method. Steps involved in this method are

Step-1	Formulation of Potential energy Equation (Π) π = Strain Energy + Work Potential (SE+WP)
Step-2	Assume a Trail function which satisfies boundary condition y = a+bx  x = 0, y= 0
Step-3	Substitute the trail function into potential energy equation
Step-4	Minimize the PE functional so as to obtain the equilibrium condition
Step-5	Solution of the system of linear algebraic equations. Π (find unknowns)
Step-6	Calculation of displacements and stresses

Problems will be solved In class room

1. 5 Rayleigh Ritz Method

Problem type	SE	WP	Π = SE + Wp	BC	Trail func
	$\frac{1}{2}\int_0 EA\left(\frac{\partial u}{\partial x}\right)^2 dx$	$-Pu_m$	$\frac{1}{2}\int_0^l EA\left(\frac{\partial u}{\partial x}\right)^2 dx - pu_m$	At x= 0 u=0 At x = L u =0	$U=a_0+a_1x +$
	$\frac{1}{2}\int_0 EA\left(\frac{\partial u}{\partial x}\right)^2 dx$	$-Pu_m$	$\frac{1}{2}\int_0^l EA\left(\frac{\partial u}{\partial x}\right)^2 dx - pu_m$	At x= 0 u=0	$U=a_0+a_1x +$
	$\frac{1}{2}\int_0^l EA\left(\frac{\partial u}{\partial x}\right)^2 dx$	$-\int_0^l uFAdx$	$\frac{1}{2}\int_0^l EA\left(\frac{\partial u}{\partial x}\right)^2 dx - \int_0^l uFAdx$	At x= 0 u=0 At x = L u =0	$U=a_0+a_1x +$
	$\frac{1}{2}\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx$	$-P y_m$	$\frac{1}{2}\int_0 EI\left(\frac{d^2y}{dx^2}\right)^2 dx - py_m$	At x= 0 y=0 At x =0 dy/dx = 0	$U=a_0+a_1x +$ $+ a_3x^3$

Problem type	SE	WP	Π = SE + Wp	BC	Trail function
		$-\int_0^l yFAdx$		At x= 0 y=0 At x =0 dy/dx = 0	$U=a_0+a_1x + a_2x^2 + a_3x^3$
	$\frac{1}{2}\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx$	$-P\, y_m$		At x= 0 y=0 At x =L, y = 0 At x =L/2 dy/dx =0	$= c_1 Sin\left(\frac{\pi x}{L}\right) + c_1 Sin\left(\frac{3\pi x}{L}\right)$
		$-\int_0^l yFAdx$		At x= 0 y=0 At x =L, y = 0 At x =L/2 dy/dx =0	$= c_1 Sin\left(\frac{\pi x}{L}\right) + c_1 Sin\left(\frac{3\pi x}{L}\right)$

1.6 Galerkin's Method

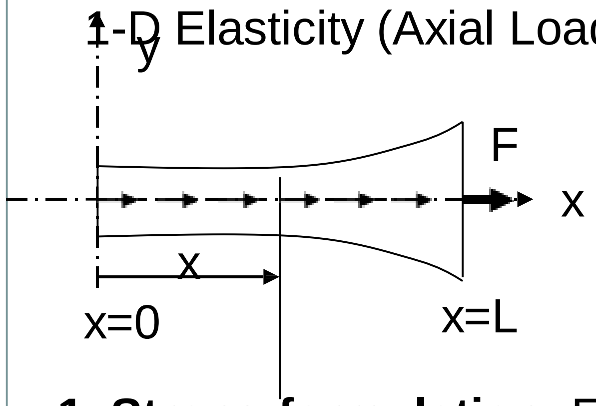
It is a method of determining approximate solution for a given problem assuming a trail function. Steps involved in this method are

Step-1	Formulate the differential Equation (DE) of equilibrium
Step-2	Assume a Trail function which satisfies boundary condition
Step-3	Substitute the displacement function into differential equation of equilibrium equation is satisfied, it not the difference due to the approximate function is denoted as "R" where R is called Residual
Step-4	Determine the constants of the function used by using the Galerkin's formula $\int_0^L f_1(x) R dx = 0$ $\int_0^L f_2(x) R dx = 0$ Where $f_1(x)$ and $f_2(x)$ are the functions of displacement models
Step-5	Knowing the constants of displacement function determine the unknown values.

Problems will be solved In class room

1.7 Basic Equation of Elasticity

1-D Elasticity (Axial Loaded Bar)



A(x) = cross section at x
b(x) = body force distribution (force / unit length)
E(x) = Young's modulus
u(x) = displacement of the bar at x

1. Strong formulation: Equilibrium equation + boundary conditions

Equilibrium equation

$$\frac{d\sigma}{dx} + b(x) = 0; \qquad 0 < x < L$$

Boundary conditions

$$u = 0 \quad at \quad x = 0$$

$$EA\frac{du}{dx} = F \quad at \ x = L$$

2. **Strain-displacement** relationship:

$$\varepsilon(x) = \frac{du}{dx}$$

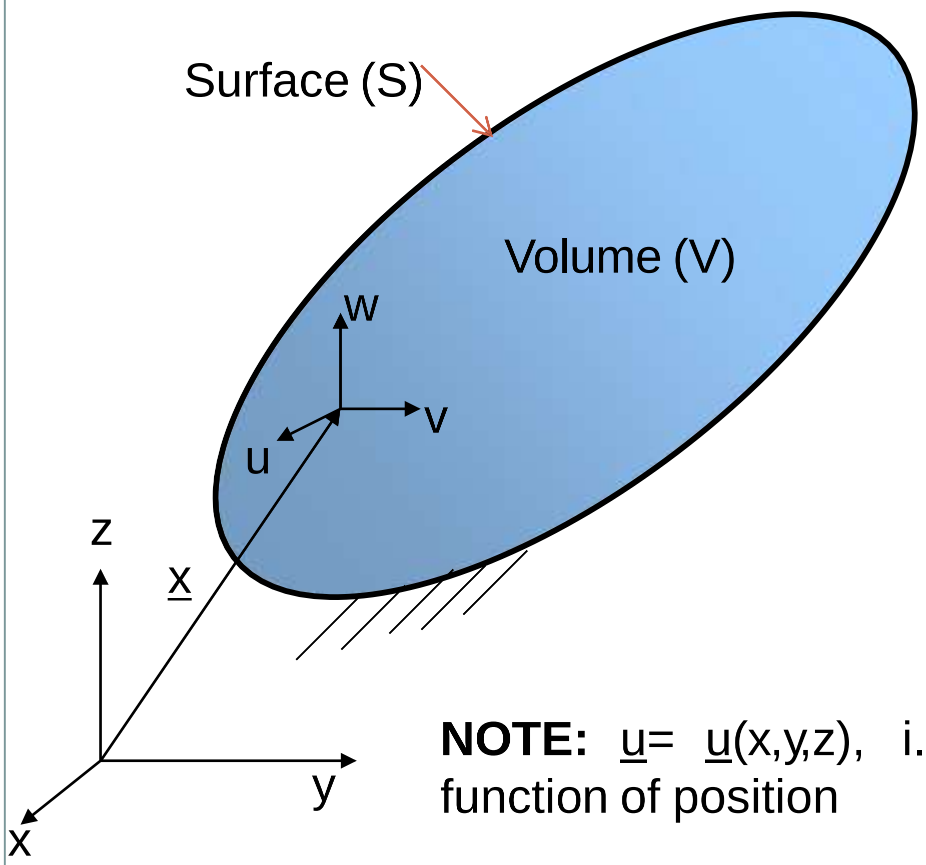
3. **Stress-strain** (constitutive) relation :

$$\sigma(x) = E\,\varepsilon(x)$$

E: Elastic (Young's) modulus of bar

1.8 Strain displacement relation

3D Elasticity



V: Volume of body, m^3
S: Total surface of the body, m^2
The deformation at point $\underline{x} = [x,y,z]^T$ is given by the 3 components of its displacement

$$\underline{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

NOTE: $\underline{u} = \underline{u}(x,y,z)$, i.e., each displacement component is a function of position

1.8 Strain displacement relation

Strain-displacement relationship in 3D (2)

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\boxed{\underline{\varepsilon} = \underline{\partial}\,\underline{u}}$$

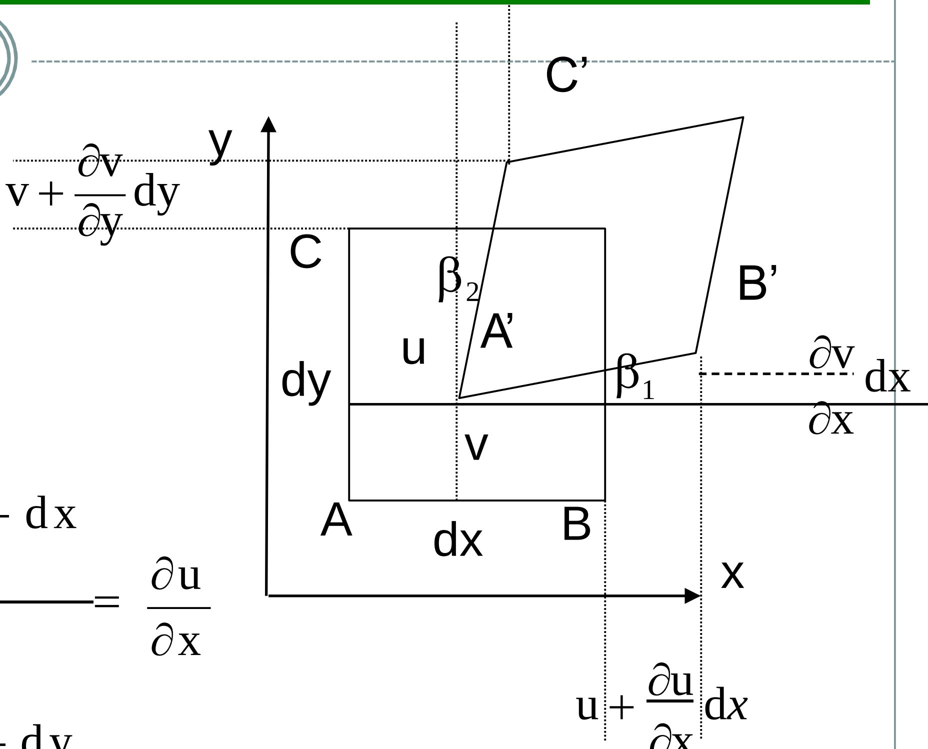
$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$\underline{\partial} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}$$

$$\underline{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

1.8 Strain displacement relation

In 2D



$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{\left(dx + \left(u + \frac{\partial u}{\partial x}dx\right) - u\right) - dx}{dx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{A'C' - AC}{AC} = \frac{\left(dy + \left(v + \frac{\partial v}{\partial y}dy\right) - v\right) - dy}{dy} = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\pi}{2} - \text{angle}\ (C'A'B') = \beta_1 + \beta_2 \approx \tan\beta_1 + \tan\beta_2$$

$$\approx \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}$$

1.9 Stress-Strain Relationship

Linear elastic material (Hooke's Law) in 3D

$$\underline{\sigma} = \underline{D}\,\underline{\varepsilon}$$

$$\underline{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

1.10 Plane stress and Strain conditions

1. **1D elastic bar :** only 1 component of the stress (stress) is nonzero. All other stress (strain) components are zero) Recall the (1) equilibrium, (2) strain-displacement and (3) stress-strain laws

2. 2D elastic problems: 2 situations

PLANE STRESS:

If a body has smaller dimensions along with the normal (longitudinal) direction (z-axis) and loading applied in this direction.

Or

Plane stress is defined to be a state of stress in which the normal stress and shear stress directed perpendicular to the plane are assumed to be zero e.g. thin plate.

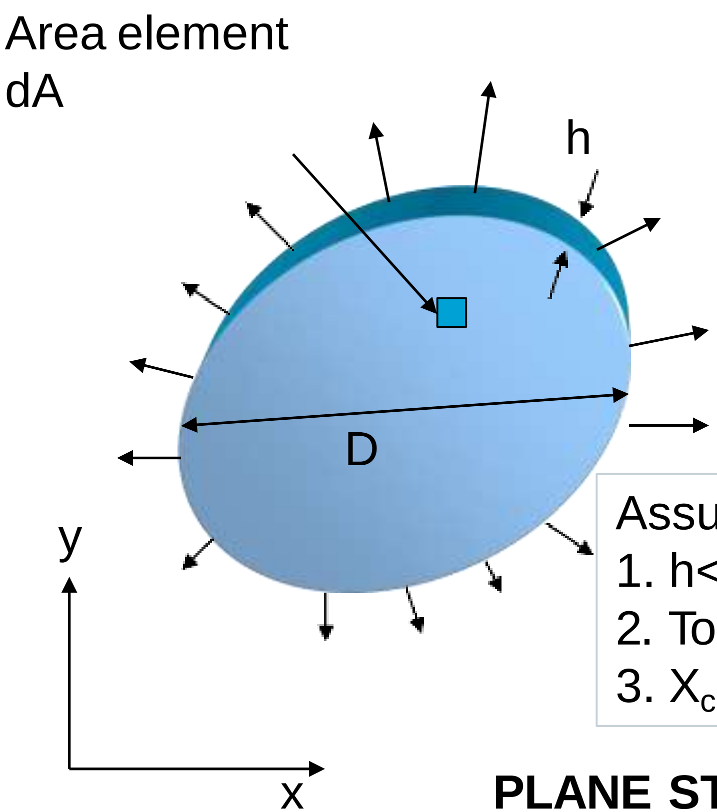
PLANE STRAIN:

If the dimensions along longitudinal direction is very long and loading subjected perpendicular to longitudinal axis

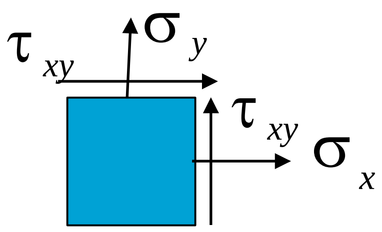
Or

Plane strain is defined to be a state of strain in which normal strain and shear strain normal to the XY plane are assumed to be zero.

1.10 Plane stress and Strain conditions



Nonzero stress components $\sigma_x, \sigma_y, \tau_{xy}$



Assumptions:

1. h<<D
2. Top and bottom surfaces are free from traction
3. X_c=0 and p_z=0

PLANE STRESS Examples: 1. Thin plate with a hole and 2. thin cantilever beam

1.10 Plane stress and Strain conditions

PLANE STRESS

Nonzero **stresses**: $\sigma_x, \sigma_y, \tau_{xy}$

Nonzero **strains**: $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}$

Isotropic linear elastic stress-strain law

$$\underline{\sigma} = \underline{D}\,\underline{\varepsilon}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right)$$

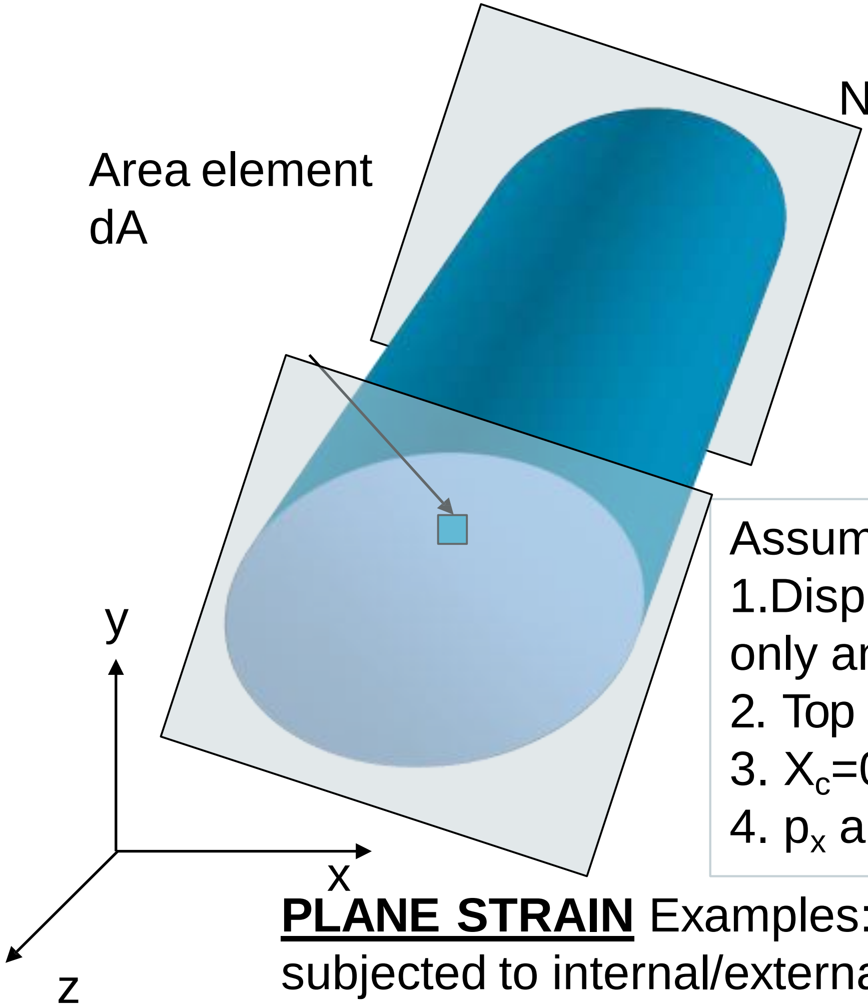
Hence, the $\underline{D}$ matrix for the **plane stress case** is

$$\underline{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

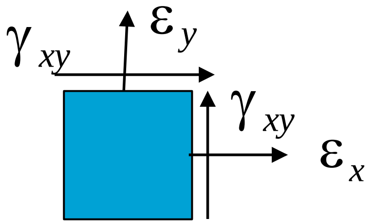
Derivation will be solved In class room

1.10 Plane stress and Strain conditions

PLANE STRAIN: Only the in-plane **strain** components are nonzero



Nonzero **strain** components $\varepsilon_x, \varepsilon_y, \gamma_{xy}$



Assumptions:

1. Displacement components u,v functions of (x,y) only and w=0
2. Top and bottom surfaces are fixed
3. X_c=0
4. p_x and p_y do not vary with z

PLANE STRAIN Examples: 1. Dam, 2 Long cylindrical pressure vessel subjected to internal/external pressure and constrained at the ends

1.10 Plane stress and Strain conditions

PLANE STRAIN

Nonzero **stress**: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}$

Nonzero **strain** components: $\varepsilon_x, \varepsilon_y, \gamma_{xy}$

Isotropic linear elastic stress-strain law

$$\underline{\sigma} = \underline{D}\,\underline{\varepsilon}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right)$$

Hence, the D matrix for the **plane strain case** is

$$\underline{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Derivation will be solved In class room

1.13 Gauss-Elimination Method

A set of n equations and n unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

(n-1) steps of forward elimination

Step 1: For Equation 2, divide Equation 1 by and multiply by .

$$\left[\frac{a_{21}}{a_{11}}\right](a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1)$$

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Subtract the result from Equation 2.

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$-a_{21}x_1 - \frac{a_{21}}{a_{11}}a_{12}x_2 - \ldots - \frac{a_{21}}{a_{11}}a_{1n}x_n = -\frac{a_{21}}{a_{11}}b_1$$

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

or

$$a'_{22}x_2 + \ldots + a'_{2n}x_n = b'_2$$

Repeat this procedure for the remaining equations to reduce the set of equations as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n = b'_3$$

$$a'_{n2}x_2 + a'_{n3}x_3 + \ldots + a'_{nn}x_n = b'_n$$

1.13 Gauss-Elimination Method

Step 2 : Repeat the same procedure for the 3rd term of Equation 3.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n &= b'_2 \\ a''_{33}x_3 + \dots + a''_{3n}x_n &= b''_3 \\ a''_{n3}x_3 + \dots + a''_{nn}x_n &= b''_n \end{aligned}$$

Step: n-1: At the end of (n-1) Forward Elimination steps, the system of equations will look like

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n &= b'_2 \\ a''_{33}x_3 + \dots + a''_{3n}x_n &= b''_3 \\ &\vdots \\ a_{nn}^{(n-1)}x_n &= b_n^{(n-1)} \end{aligned}$$

Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdot & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdot & a'_{2n} \\ 0 & 0 & a''_{33} & \cdot & a''_{3n} \\ \vdots & \vdots & \vdots & \cdot & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

Back Substitution

Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \quad \text{for } i = n-1, \dots, 1$$

Problems will be solved In class room